CHAPTER 7

Partial Differentiation

From the previous two chapters we know how to differentiate **functions of one variable**. But many functions in economics depend on several variables: output depends on both capital and labour, for example. In this chapter we show how to differentiate **functions of several variables** and apply this to find the **marginal products of labour and capital, marginal utilities, and elasticities of demand**. We use **differentials** to find the gradients of isoquants, and hence determine **marginal rates of substitution**.

1. Partial Derivatives

The derivative of a function of one variable, such as $y(x)$, tells us the gradient of the function: how y changes when x increases. If we have a function of more than one variable, such as:

$$z(x,y) = x^3 + 4xy + 5y^2$$

we can ask, for example, how z changes when x increases but y doesn't change. The answer to this question is found by thinking of z as a function of x, and differentiating, treating y as if it were a constant parameter:

$$\frac{\partial z}{\partial x} = 3x^2 + 4y$$

This process is called partial differentiation. We write $\frac{\partial z}{\partial x}$ rather than $\frac{dz}{dx}$, to emphasize that z is a function of another variable as well as x, which is being held constant.

$\dfrac{\partial z}{\partial x}$ is called the partial derivative of z with respect to x

$\frac{\partial z}{\partial x}$ is pronounced "partial dee z by dee x".

Similarly, if we hold x constant, we can find the partial derivative with respect to y:

$$\frac{\partial z}{\partial y} = 4x + 10y$$

Remember from Chapter 4 that you can think of $z(x,y)$ as a "surface" in 3 dimensions. Imagine that x and y represent co-ordinates on a map, and $z(x,y)$ represents the height of the land at the point (x,y). Then, $\frac{\partial z}{\partial x}$ tells you the gradient of the land as you walk in the direction of increasing x, keeping the y co-ordinate constant. If $\frac{\partial z}{\partial x} > 0$, you are walking uphill; if it is negative you are going down. (Try drawing a picture to illustrate this.)

Exercises 7.1: Find the partial derivatives with respect to x and y of the functions:

(1) $f(x, y) = 3x^2 - xy^4$

(2) $h(x, y) = (x + 1)^2(y + 2)$

(3) $g(x, y) = \dfrac{\ln x}{y}$

Examples 1.1: For the function $f(x, y) = x^3 e^{-y}$:

(i) Show that f is increasing in x for all values of x and y.

$$\frac{\partial f}{\partial x} = 3x^2 e^{-y}$$

Since $3x^2 > 0$ for all values of x, and $e^{-y} > 0$ for all values of y, this derivative is always positive: f is increasing in x.

(ii) For what values of x and y is the function increasing in y?

$$\frac{\partial f}{\partial y} = -x^3 e^{-y}$$

When $x < 0$ this derivative is positive, so f is increasing in y. When $x > 0$, f is decreasing in y.

1.1. Second-order Partial Derivatives

For the function in the previous section:

$$z(x, y) = x^3 + 4xy + 5y^2$$

we found:

$$\frac{\partial z}{\partial x} = 3x^2 + 4y$$
$$\frac{\partial z}{\partial y} = 4x + 10y$$

These are the *first-order* partial derivatives. But we can differentiate again to find *second-order* partial derivatives. The second derivative with respect to x tells us how $\frac{\partial z}{\partial x}$ changes as x increases, still keeping y constant.

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

Similarly:

$$\frac{\partial^2 z}{\partial y^2} = 10$$

We can also differentiate $\frac{\partial z}{\partial x}$ with respect to y, to find out how it changes when y increases. This is written as $\frac{\partial^2 z}{\partial y \partial x}$ and is called a *cross-partial* derivative:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 4$$

Similarly:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 4$$

Note that the two cross-partial derivatives are the same: it doesn't matter whether we differentiate with respect to x first and then with respect to y, or vice-versa. This happens with all "well-behaved" functions.

A function $z(x, y)$ of two variables has four second-order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 z}{\partial y \partial x}$$

$$\text{but } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

1.2. Functions of More Than Two Variables

A function of n variables:

$$f(x_1, x_2, \ldots, x_n)$$

has n first-order partial derivatives:

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$$

and n^2 second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_2 \partial x_1}, \frac{\partial^2 f}{\partial x_3 \partial x_1}, \ldots$$

Again we find that it doesn't matter which variable we differentiate with respect to first: the cross-partials are equal. For example:

$$\frac{\partial^2 f}{\partial x_3 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_3}$$

1.3. Alternative Notation

There are several different ways of writing partial derivatives. For a function $f(x, y)$ the first-order partial derivatives can be written:

$$\frac{\partial f}{\partial x} \text{ or } f_x \quad \text{and} \quad \frac{\partial f}{\partial y} \text{ or } f_y$$

and the second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{xy}$$

For a function $f(x_1, x_2, x_3, \ldots)$ we sometimes write:

$$f_1 \text{ for } \frac{\partial f}{\partial x_1}, \quad f_2 \text{ for } \frac{\partial f}{\partial x_2}, \quad f_{11} \text{ for } \frac{\partial^2 f}{\partial x_1^2}, \quad f_{13} \text{ for } \frac{\partial^2 f}{\partial x_1 \partial x_3} \text{ etc.}$$

EXERCISES 7.2:

(1) Find all the first- and second-order partial derivatives of the function $f(x, y) = x^2 + 3yx$, verifying that the two cross-partial derivatives are the same.

(2) Find all the first- and second-order partial derivatives of $g(p, q, r) = q^2 e^{2p+1} + rq$.

(3) If $z(x, y) = \ln(2x + 3y)$, where $x > 0$ and $y > 0$, show that z is increasing in both x and y.

(4) For the function $f(x, y, z) = (x^2 + y^2 + z^2)^3$ find the partial derivatives f_x, f_{xx}, and f_{xz}.

(5) If $F(K, L) = AK^\alpha L^\beta$, show that $F_{KL} = F_{LK}$.

(6) For the utility function $u(x_1, x_2, x_3) = x_1 + 2x_2x_3$, find the partial derivatives u_{23} and u_{12}.

Further Reading and Exercises

- *Jacques* §5.1
- *Anthony & Biggs* §11.1 and §11.2

2. Economic Applications of Partial Derivatives, and Euler's Theorem

2.1. The Marginal Products of Labour and Capital

Suppose that the output produced by a firm depends on the amounts of labour and capital used. If the production function is

$$Y(K, L)$$

the partial derivative of Y with respect to L tells us the the marginal product of labour:

$$MPL = \frac{\partial Y}{\partial L}$$

The marginal product of labour is the amount of extra output the firm could produce if it used one extra unit of labour, but kept capital the same as before.
Similarly the marginal product of capital is:

$$MPK = \frac{\partial Y}{\partial K}$$

EXAMPLES 2.1: For a firm with production function $Y(K, L) = 5K^{\frac{1}{3}}L^{\frac{2}{3}}$:

(i) Find the marginal product of labour.

$$MPL = \frac{\partial Y}{\partial L} = \frac{10}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}$$

(ii) What is the MPL when $K = 64$ and $L = 125$?

$$MPL = \frac{10}{3}(64)^{\frac{1}{3}}(125)^{-\frac{1}{3}} = \frac{10}{3} \times 4 \times \frac{1}{5} = \frac{8}{3}$$

(iii) What happens to the marginal product of labour as the number of workers increases?

Differentiate MPL with respect to L: $\dfrac{\partial^2 Y}{\partial L^2} = -\dfrac{10}{9}K^{\frac{1}{3}}L^{-\frac{4}{3}} < 0$

So the MPL decreases as the labour input increases – the firm has diminishing returns to labour, if capital is held constant. This is true for *all* values of K and L.

(iv) What happens to the marginal product of labour as the amount of capital increases?

Differentiate MPL with respect to K: $\dfrac{\partial^2 Y}{\partial K \partial L} = \dfrac{10}{9}K^{-\frac{2}{3}}L^{-\frac{2}{3}} > 0$

So the MPL increases as the capital input increases – when there is more capital, an additional worker is more productive.

2.2. Elasticities of Demand

If a consumer has a choice between two goods, his demand x_1 for good 1 depends on its price, p_1, but also on the consumer's income, m, and the price of the other good. The demand function can by written:

$$x_1(p_1, p_2, m)$$

The *own-price elasticity of demand* is the responsiveness of demand to changes in the price p_1, defined just as in Chapter 6 except that we now have to use the partial derivative:

$$\epsilon_{11} = \frac{p_1}{x_1}\frac{\partial x}{\partial p_1}$$

But in addition we can measure the responsiveness of the demand for good 1 to changes in the price of good 2, or the consumer's income:

For the demand function $x_1(p_1, p_2, m)$

- The own-price elasticity is: $\epsilon_{11} = \dfrac{p_1}{x_1}\dfrac{\partial x_1}{\partial p_1}$
- The cross-price elasticity is: $\epsilon_{12} = \dfrac{p_2}{x_1}\dfrac{\partial x_1}{\partial p_2}$
- The income elasticity is: $\eta_1 = \dfrac{m}{x_1}\dfrac{\partial x_1}{\partial m}$

2.3. Euler's Theorem

Remember from Chapter 4 that a function of two variables is called *homogeneous* of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. If a production function is homogeneous of degree 1, for example, it has constant returns to scale. Euler's Theorem states that:

If the function $f(x, y)$ is homogeneous of degree n:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

Examples 2.2: $f(x, y) = 2x^2y^2 + xy^3$

(i) Show that f is homogenous.

$$\begin{aligned} f(\lambda x, \lambda y) &= 2(\lambda x)^2(\lambda y)^2 + \lambda x(\lambda y)^3 \\ &= 2\lambda^4 x^2y^2 + \lambda^4 xy^3 \\ &= \lambda^4 f(x, y) \end{aligned}$$

(ii) Verify that it satisfies Euler's Theorem.

$$\frac{\partial f}{\partial x} = 4xy^2 + y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x^2y + 3xy^2$$

$$\begin{aligned} \Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= 4x^2y^2 + xy^3 + 4x^2y^2 + 3xy^3 \\ &= 8x^2y^2 + 4xy^3 \\ &= 4f(x, y) \end{aligned}$$

Exercises 7.3: Economic Applications of Partial Derivatives

(1) For the production function $Q(K, L) = (K-2)L^2$:
   (a) Find the marginal product of labour when $L = 4$ and $K = 5$.
   (b) Find the marginal product of capital when $L = 3$ and $K = 4$.
   (c) How does the marginal product of capital change as labour increases?

(2) Show that the production function $Y = (K^{\frac{1}{2}} + L^{\frac{1}{2}})^2$ has constant returns to scale. Verify that it satisfies Euler's Theorem.

(3) For the demand function $x_1(p_1, p_2, m) = \dfrac{p_2 m}{p_1^2}$
   (a) Find the own-price, cross-price, and income elasticities.
   (b) Show that the function is homogeneous of degree zero. Why is it economically reasonable to expect a demand function to have this property?

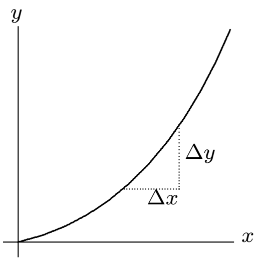
(4) Write down Euler's Theorem for a demand function $x_1(p_1, p_2, m)$ that is homogeneous of degree zero. What does this tell you about the own-price, cross-price, and income elasticities?

Further Reading and Exercises

- *Jacques* §5.2
- *Anthony and Biggs* §12.4

3. Differentials

3.1. Derivatives and Approximations: Functions of One Variable



If we have a function of one variable $y(x)$, and x increases by a small amount Δx, how much does y change? We know that when Δx is small:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

Rearranging:

$$\Delta y \approx \frac{dy}{dx}\Delta x$$

Examples 3.1: If we know that the marginal propensity to consume is 0.85, and national income increases by £2bn, then from $\Delta C \approx \frac{dC}{dY}\Delta Y$, aggregate consumption will increase by $0.85 \times 2 = £1.7$bn.

3.2. Derivatives and Approximations: Functions of Several Variables

Similarly, if we have a function of two variables, $z(x, y)$, and x increases by a small amount Δx, but y doesn't change:

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x$$

But if both x and y change, we can calculate the change in z from:

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

Examples 3.2: If the marginal product of labour is 1.5, and the marginal product of capital is 1.8, how much will a firm's output change if it employs one more unit of capital, and one fewer worker?

$$\Delta Y \approx \frac{\partial Y}{\partial L}\Delta L + \frac{\partial Y}{\partial K}\Delta K$$

Putting: $\frac{\partial Y}{\partial L} = 1.5$, $\frac{\partial Y}{\partial K} = 1.8$, $\Delta K = 1$, and $\Delta L = -1$, we obtain: $\Delta Y = 0.3$.

3.3. Differentials

The approximation

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

becomes more accurate as the changes in the arguments Δx and Δy beome smaller. In the limit, as they become *infinitesimally* small, we write dx and dy to represent infinitesimal changes in x and y. They are called *differentials*. The corresponding change in z, dz, is called the *total differential*.

> The total differential of the function $z(x, y)$ is
> $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Examples 3.3: Find the total differential of the following functions:

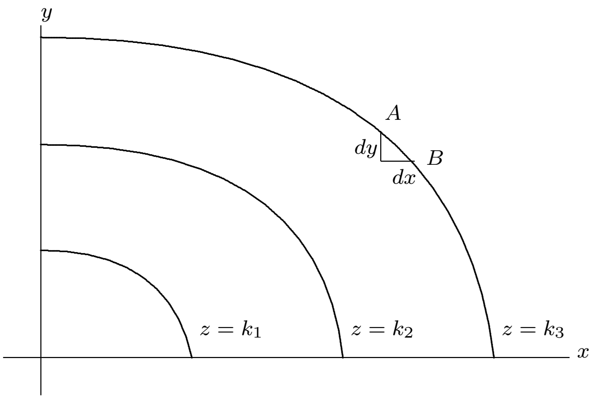
(i) $F(x,y) = x^4 + xy^3$

$$\frac{\partial F}{\partial x} = 4x^3 + y^3 \text{ and } \frac{\partial F}{\partial y} = 3xy^2$$
$$\Rightarrow dF = (4x^3 + y^3)dx + 3xy^2 dy$$

(ii) $u(x_1, x_2, x_3) = 2x_1 + 5x_2 + 9x_3$

$$du = 2dx_1 + 5dx_2 + 9dx_3$$

3.4. Using Differentials to Find the Gradient of an Isoquant



Remember from Chapter 4 that we can represent a function $z(x,y)$ by drawing the isoquants: the lines

$$z(x,y) = k$$

for different values of the constant k.

To find the gradient at the point A, consider a small change in x and y that takes you to a point B on the same isoquant.

Then the change in z is given by: $\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$

But in moving from A to B, z does not change, so $dz = 0$: $\quad \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = 0$

Rearranging this equation gives us $\frac{dy}{dx}$:

The gradient of the isoquant of the function $z(x,y)$ at any point (x,y) is given by:

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

Examples 3.4: The diagram above shows the function $z = x^2 + 2y^2$.

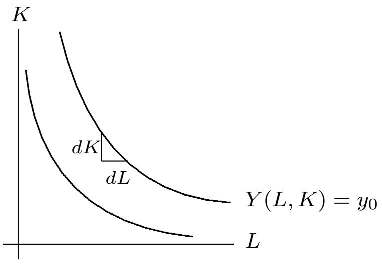
(i) Which isoquant passes through the point where $x = 2$ and $y = 1$?
At (2,1), $z = 6$, so the isoquant is $x^2 + 2y^2 = 6$.

(ii) Find the gradient of the isoquant at this point.

$$\begin{aligned} dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy &= 0 \\ \Rightarrow 2xdx + 4ydy &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{x}{2y} \end{aligned}$$

So at (2,1) the gradient is -1.

3.5. Economic Application: The Marginal Rate of Technical Substitution



For a production function $Y(L, K)$, the isoquants show the combinations of labour and capital that can be used to produce the same amount of output.

At any point (L, K), the slope of the isoquant is the *Marginal Rate of Technical Substitution* between labour and capital.

The $MRTS$ tells you how much less capital you need, to produce the same quantity of output, if you increase labour by a small amount dL.

Using the same method as in the previous section, we can say that along the isoquant output doesn't change:

$$\begin{aligned} dY = \frac{\partial Y}{\partial L}dL + \frac{\partial Y}{\partial K}dK &= 0 \\ \Rightarrow \frac{dK}{dL} &= -\frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial K}} \\ \Rightarrow MRTS &= -\frac{MPL}{MPK} \end{aligned}$$

The Marginal Rate of Technical Substitution between two inputs to production is the (negative of the) ratio of their marginal products.

EXERCISES 7.4: **Differentials and Isoquants**

(1) A firm is producing 1550 items per week, and its total weekly costs are £12000. If the marginal cost of producing an additional item is £8, estimate the firm's total costs if it increases weekly production to 1560 items.

(2) A firm has marginal product of labour 3.4, and marginal product of capital 2.0. Estimate the change in output if reduces both labour and capital by one unit.

(3) Find the total differential for the following functions:

   (a) $z(x, y) = 10x^3y^5$

   (b) $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2x_3 - 5x_4^2 + 7$

(4) For the function $g(x, y) = x^4 + x^2y^2$, show that slope of the isoquant at the point (x, y) is $-\dfrac{2x^2 + y^2}{xy}$. What is slope of the indifference curve $g = 10$ at the point where $x = 1$?

(5) Find the Marginal Rate of Technical Substitution for the production function $Y = K(L^2 + L)$.

Further Reading and Exercises

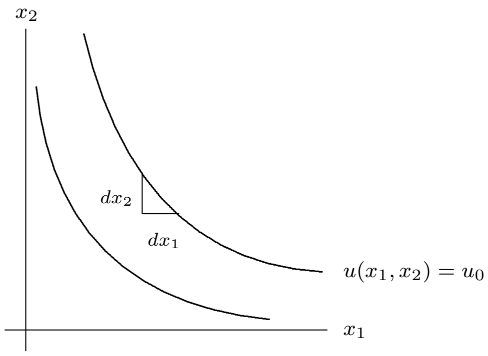
- *Jacques* §5.1 and §5.2.

4. Economic Application: Utility and Indifference Curves

If a consumer has well-behaved preferences over two goods, good 1 and good 2, his preferences can be represented by a utility function $u(x_1, x_2)$.

Then the marginal utilities of good 1 and good 2 are given by the partial derivatives:

$$MU_1 = \frac{\partial u}{\partial x_1} \text{ and } MU_2 = \frac{\partial u}{\partial x_2}$$



His indifference curves are the isoquants of u.

At any point (x_1, x_2), the slope of the isoquant is the *Marginal Rate of Substitution* between good 1 and good 2.

$$MRS = \frac{dx_2}{dx_1}$$

The MRS is the rate at which he is willing to substitute good 2 for good 1, remaining on the same indifference curve.

As before, we can find the marginal rate of substitution at any bundle (x_1, x_2) by taking the total differential of utility:

$$\begin{aligned} du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 &= 0 \\ \Rightarrow \frac{dx_2}{dx_1} &= -\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} \\ \Rightarrow MRS &= -\frac{MU_1}{MU_2} \end{aligned}$$

The Marginal Rate of Substitution between two goods is the (negative of the) ratio of their marginal utilities.

4.1. Perfect Substitutes

For the utility function: $u(x_1, x_2) = 3x_1 + x_2$
we find:

$$MU_1 = 3 \text{ and } MU_2 = 1 \text{ so } MRS = -3$$

The MRS is constant and negative, so the indifference curves are downward-sloping straight lines. The consumer is always willing to substitute 3 units of good 2 for a unit of good 1, whatever bundle he is consuming. The goods are perfect substitutes.

Any utility function of the form $u(x_1, x_2) = ax_1 + bx_2$, where a and b are positive parameters, represents perfect substitutes.

4.2. Cobb-Douglas Utility

For the utility function

$$u(x_1, x_2) = x_1^a x_2^b$$

where a and b are positive parameters, we find:

$$MU_1 = a x_1^{a-1} x_2^b \text{ and } MU_2 = b x_1^a x_2^{b-1}$$

$$\Rightarrow MRS = -\frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = -\frac{a x_2}{b x_1}$$

From this expression we can see:

- The MRS is negative so the indifference curves are downward-sloping.
- As you move along an indifference curve, increasing x_1 and decreasing x_2, the MRS becomes less negative, so the indifference curve becomes flatter. Hence the indifference curves are convex (as in the diagram on the previous page).

4.3. Transforming the Utility Function

If a consumer has a preference ordering represented by the utility function $u(x_1, x_2)$, and $f(u)$ is a monotonic increasing function (that is, $f'(u) > 0$ for all u) then $v(x_1, x_2) = f(u(x_1, x_2))$ is a utility function representing the same preference ordering, because whenever a bundle (x_1, x_2) is preferred to (y_1, y_2):

$$u(x_1, x_2) > u(y_1, y_2) \Leftrightarrow f(u(x_1, x_2)) > f(u(y_1, y_2))$$

The two utility functions represent the same preferences, provided that we are only interested in the consumer's *ordering* of bundles, and don't attach any significance to the utility numbers.

EXAMPLES 4.1: If a consumer has a Cobb-Douglas utility function:

$$u(x_1, x_2) = x_1^a x_2^b$$

$f(u) = \ln u$ is a monotonic increasing function so his preferences can also be represented by

$$v(x_1, x_2) = \ln\left(x_1^a x_2^b\right) = a \ln x_1 + b \ln x_2$$

Note that if we calculate the MRS for the function v we find:

$$MU_1 = \frac{a}{x_1} \text{ and } MU_2 = \frac{b}{x_2} \quad \text{so} \quad MRS = -\frac{a x_2}{b x_1}$$

Although their marginal utilities are different, v has the same MRS as u.

EXERCISES 7.5: Utility and Indifference curves

(1) Find the MRS for each of the following utility functions, and hence determine whether the indifference curves are convex, concave, or straight:

(a) $u = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$ (b) $u = x_1^2 + x_2^2$ (c) $u = \ln(x_1 + x_2)$

(2) Prove that for any monotonic increasing function $f(u)$, the utility function $v(x_1, x_2) = f(u(x_1, x_2))$ has the same MRS as the utility function u.
(*Hint*: Use the Chain Rule to differentiate v.)

Further Reading and Exercises

- *Jacques* §5.2.
- *Varian* "Intermediate Microeconomics" Chapter 4

5. The Chain Rule and Implicit Differentiation

5.1. The Chain Rule for Functions of Several Variables

If z is a function of two variables, x and y, and both x and y depend on another variable, t (time, for example), then z also depends on t. We have:

If $z = z(x, y)$, and x and y are functions of t, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Note the similarity of this rule to

- the Chain Rule for functions of one variable in Chapter 6, and
- the formula for the Total Differential in section 3.3

Examples 5.1: Suppose the aggregate output of a country is given by:

$$Y = AK^\alpha L^\beta$$

where K is the capital stock, and L is the labour force. If the capital stock and the labour force are each growing according to:

$$L(t) = L_0 e^{nt} \text{ and } K(t) = K_0 e^{mt}$$

(i) What are the proportional growth rates of capital and labour?
As in Chapter 6 we have:

$$\frac{dL}{dt} = nL_0 e^{nt} = nL(t) \text{ and similarly for } K(t)$$

So labour is growing at a constant proportional rate n, and capital is growing at constant proportional rate m.

(ii) What is the rate of growth of output?
Applying the Chain Rule:

$$\begin{aligned}
\frac{dY}{dt} &= \frac{\partial Y}{\partial K}\frac{dK}{dt} + \frac{\partial Y}{\partial L}\frac{dL}{dt} \\
\Rightarrow \frac{dY}{dt} &= \alpha AK^{\alpha-1}L^\beta \times mK + \beta AK^\alpha L^{\beta-1} \times nL \\
&= \alpha m AK^\alpha L^\beta + \beta n AK^\alpha L^\beta \\
&= (\alpha m + \beta n)Y
\end{aligned}$$

So output grows at a constant proportional rate $(\alpha m + \beta n)$.

5.2. Implicit Differentiation

The equation:

$$x^2 + 2xy + y^3 - 4 = 0$$

describes a relationship between x and y. If we plotted values of x and y satisfying this equation, they would form a curve in (x, y) space. We cannot easily rearrange the equation to find y as an *explicit* function of x, but we can think of y as an *implicit* function of x, and ask how y changes when x changes – or in other words, what is $\frac{dy}{dx}$? To answer this question, we can think of the left-hand side of the equation

$$x^2 + 2xy + y^3 - 4 = 0$$

as a function of two variables, x and y. And in general, if we have any equation:

$$f(x, y) = 0$$

we can find the gradient of the relationship between x and y in exactly the same way as we found the gradient of an isoquant in section 3.4, by totally differentiating:

$$\begin{aligned} df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy &= 0 \\ \Rightarrow \frac{\partial f}{\partial y}dy &= -\frac{\partial f}{\partial x}dx \\ \Rightarrow \frac{dy}{dx} = &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \end{aligned}$$

Finding the gradient of $y(x)$ when y is an implicit function of x is called *implicit differentiation*. The method explained here, using the total derivative, is the same as the one in *Jacques*. *Anthony and Biggs* use an alternative (but equivalent) method, applying the Chain Rule of the previous section.

EXAMPLES 5.2: *Implicit Differentiation*

(i) Find $\frac{dy}{dx}$ if $\quad x^2 + 5xy + y^3 - 4 = 0$

$$\begin{aligned} (2x + 5y)dx + (5x + 3y^2)dy &= 0 \\ (5x + 3y^2)dy &= -(2x + 5y)dx \\ \frac{dy}{dx} = &= -\frac{2x + 5y}{5x + 3y^2} \end{aligned}$$

(ii) Find $\frac{dq}{dp}$ if $\quad q^2 + e^{3p} - pe^{2q} = 0$

$$\begin{aligned} (2q - 2pe^{2q})dq + (3e^{3p} - e^{2q})dp &= 0 \\ (2q - 2pe^{2q})dq &= (e^{2q} - 3e^{3p})dp \\ \frac{dq}{dp} = &= \frac{e^{2q} - 3e^{3p}}{2(q - pe^{2q})} \end{aligned}$$

EXERCISES 7.6: **The Chain Rule and Implicit Differentiation**

(1) If $z(x, y) = 3x^2 + 2y^3$, $x = 2t + 1$, and $y = 4t - 2$, use the chain rule to find $\frac{dz}{dt}$ when $t = 2$.

(2) If the aggregate production function is $Y = K^{\frac{1}{3}}L^{\frac{2}{3}}$, what is the marginal product of labour? If the labour force grows at a constant proportional rate n, and the capital stock grows at constant proportial rate m, find the proportional growth rate of (a) output, Y and (b) the MPL.

(3) If $xy^2 + y - 2x = 0$, find $\dfrac{dy}{dx}$ when $y = 2$.

Further Reading and Exercises

- *Jacques* §5.1
- *Anthony and Biggs* §11.3, §12.1, §12.2 and §12.3.

6. Comparative Statics

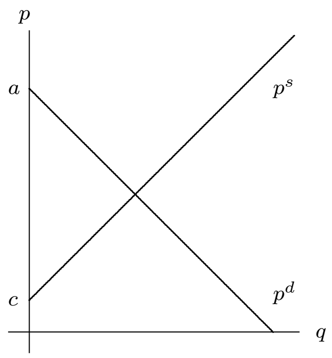
EXAMPLES 6.1:

(i) Suppose the inverse demand and supply functions for a good are:

$$p^d = a - q \quad \text{and} \quad p^s = bq + c \quad \text{where } b > 0 \text{ and } a > c > 0$$

How does the quantity sold change (a) if a increases (b) if b increases?

Putting $p^d = p^s$ and solving for the equilibrium quantity: $\quad q^* = \dfrac{a-c}{1+b}$



We can answer the question by differentiating with respect to the parameters:

$$\frac{\partial q^*}{\partial a} = \frac{1}{1+b} > 0$$

This tells us that if a increases – that is, there is an upward shift of the demand function – the equilibrium quantity will increase. (You can, of course, see this from the diagram.)

Similarly $\dfrac{\partial q^*}{\partial b} < 0$, so if the supply function gets steeper q^* decreases.

(ii) Suppose the inverse demand and supply functions for a good are:

$$p^d = a + f(q) \quad \text{and} \quad p^s = g(q) \quad \text{where } f'(q) < 0 \text{ and } g'(q) > 0$$

How does the quantity sold change if a increases?

Here we don't know much about the supply and demand functions, except that they slope up and down in the usual way. The equilibrium quantity q^* satisfies:

$$a + f(q^*) = g(q^*)$$

We cannot solve explicitly for q^*, but this equation defines it as an implicit function of a. Hence we can use implicit differentiation:

$$da + f'(q^*)dq^* = g'(q^*)dq^* \quad \Rightarrow \quad \frac{dq*}{da} = \frac{1}{g'(q^*) - f'(q^*)} > 0$$

Again, this tells us that q^* increases when the demand function shifts up.

The process of using derivatives to find out how an equilibrium depends on the parameters is called *comparative statics*.

EXERCISES 7.7: If the labour supply and demand functions are

$$l^s = aw \text{ and } l^d = \frac{k}{w} \quad \text{where } a \text{ and } k \text{ are positive parameters:}$$

(1) What happens to the equilibrium wage and employment if k increases?

(2) What happens to the equilibrium wage and employment if a *decreases*?

Further Reading and Exercises

- *Jacques* §5.3 gives some macroeconomic applications of Comparative Statics.

Solutions to Exercises in Chapter 7

EXERCISES 7.1:

(1) $\frac{\partial f}{\partial x} = 6x - y^4$ $\frac{\partial f}{\partial y} = -4xy^3$

(2) $\frac{\partial h}{\partial x} = 2(x+1)(y+2)$
$\frac{\partial h}{\partial y} = (x+1)^2$

(3) $\frac{\partial g}{\partial x} = \frac{1}{xy}$ $\frac{\partial g}{\partial y} = -\frac{\ln x}{y^2}$

EXERCISES 7.2:

(1) $f_x = 2x + 3y$, $f_y = 3x$,
$f_{xx} = 2$, $f_{yy} = 0$,
$f_{yx} = f_{xy} = 3$.

(2) $g_p = 2q^2 e^{2p+1}$,
$g_q = 2qe^{2p+1} + r$,
$g_r = q$,
$g_{pp} = 4q^2 e^{2p+1}$,
$g_{qq} = 2e^{2p+1}$,
$g_{rr} = 0$,
$g_{qp} = g_{pq} = 4qe^{2p+1}$,
$g_{rp} = g_{pr} = 0$,
$g_{rq} = g_{qr} = 1$.

(3) $\frac{\partial z}{\partial x} = \frac{2}{2x+3y} > 0$
$\frac{\partial z}{\partial y} = \frac{3}{2x+3y} > 0$

(4) $f_x = 6x(x^2 + y^2 + z^2)^2$
$f_{xx} = 6(x^2 + y^2 + z^2)^2 + 24x^2(x^2 + y^2 + z^2)$
$= 6(x^2 + y^2 + z^2)$
$\times (5x^2 + y^2 + z^2)$
$f_{xz} = 24xz(x^2 + y^2 + z^2)$

(5) $F_K = \alpha A K^{\alpha-1} L^{\beta}$
$F_L = \beta A K^{\alpha} L^{\beta-1}$
$F_{KL} = F_{LK}$
$= \alpha\beta A K^{\alpha-1} L^{\beta-1}$

(6) $u_{23} = 2$, $u_{12} = 0$

EXERCISES 7.3:

(1) (a) $\frac{\partial Q}{\partial L} = 2L(K-2)$
$= 24$
(b) $\frac{\partial Q}{\partial K} = L^2 = 9$
(c) It increases:
$\frac{\partial^2 Q}{\partial L \partial K} = 2L > 0$

(2) $Y(\lambda K, \lambda L)$
$= (\lambda^{\frac{1}{2}} K^{\frac{1}{2}} + \lambda^{\frac{1}{2}} L^{\frac{1}{2}})^2$
$= (\lambda^{\frac{1}{2}} (K^{\frac{1}{2}} + L^{\frac{1}{2}})^2$
$= \lambda(K^{\frac{1}{2}} + L^{\frac{1}{2}})^2$
$= \lambda Y(K, L)$
$F_K = K^{-\frac{1}{2}}(K^{\frac{1}{2}} + L^{\frac{1}{2}})$
$F_L = L^{-\frac{1}{2}}(K^{\frac{1}{2}} + L^{\frac{1}{2}})$
$\Rightarrow KF_K + LF_L$
$= (K^{\frac{1}{2}} + L^{\frac{1}{2}})^2$

(3) (a) $\epsilon_{11} = -2$, $\epsilon_{12} = 1$,
$\eta_1 = 1$
(b) $x_1(\lambda p_1, \lambda p_2, \lambda p_m)$
$= \frac{\lambda p_2 \lambda m}{\lambda^2 p_1^2}$
$= \frac{p_2 m}{p_1^2}$
If all prices double, but income doubles too, demands shouldn't change.

(4) $p_1 \frac{\partial x_1}{\partial p_1} + p_2 \frac{\partial x_1}{\partial p_2} + m \frac{\partial x_1}{\partial m} = 0$
Dividing by x_1 gives:
$\epsilon_{11} + \epsilon_{12} + \eta_1 = 0$

EXERCISES 7.4:

(1) $\Delta C = \frac{dC}{dq} \Delta q = 8 \times 10$
$= 80$
$\Rightarrow C = £12000 + £80$
$= £12080$

(2) $\Delta Y = 3.4 \times -1 + 2 \times -1$
$= -5.4$

(3) (a) $dz = 30x^2 y^5 dx$
$+ 50x^3 y^4 dy$
(b) $df = 2x_1 dx_1$
$+ x_3 dx_2 + x_2 dx_3$
$- 10x_4 dx_4$

(4) $(4x^3 + 2xy^2)dx$
$+ 2x^2 y dy = 0$
$\Rightarrow \frac{dy}{dx} = -\frac{4x^3 + 2xy^2}{2x^2 y}$
$= -\frac{2x^2 + y^2}{xy}$
When $x = 1$ and $g = 10$, $y = 3$.
Then $\frac{dy}{dx} = -\frac{11}{3}$

(5) $MRTS = -\frac{2KL+K}{L^2+L}$
$= -\frac{K(2L+1)}{L(L+1)}$

EXERCISES 7.5:

(1) (a) $MRS = -(\frac{x_2}{x_1})^{\frac{1}{2}}$
Convex
(b) $MRS = -(\frac{x_1}{x_2})$
Concave
(c) $MRS = -1$
Straight

(2) $\frac{\partial v}{\partial x_1} = f'(u) \frac{\partial u}{\partial x_1}$ and
$\frac{\partial v}{\partial x_2} = f'(u) \frac{\partial u}{\partial x_2}$
so the ratio of the MUs is the same for u and v.

EXERCISES 7.6:

(1) $\frac{dz}{dt} = 6x \times 2 + 6y^2 \times 4$
$= 6 \times 5 \times 2 + 6 \times 36 \times 4 = 924$

(2) $MPL = \frac{2}{3} K^{\frac{1}{3}} L^{-\frac{1}{3}}$
(a) $\frac{1}{3}m + \frac{2}{3}n$
(b) $\frac{1}{3}m - \frac{1}{3}n$

(3) When $y = 2$, $x = -1$.
$\frac{dy}{dx} = \frac{2-y^2}{2xy+1} = \frac{2}{3}$

EXERCISES 7.7:

(1) In equilibrium, $w = \sqrt{\frac{k}{a}}$ and $l = \sqrt{ak}$.
$\frac{\partial w}{\partial k} = \frac{1}{2\sqrt{ak}} > 0$ and
$\frac{\partial l}{\partial k} = \frac{1}{2}\sqrt{\frac{a}{k}} > 0$
so both w and l increase when k increases.

(2) $\frac{\partial w}{\partial a} = -\frac{1}{2}\sqrt{k} a^{-\frac{3}{2}} < 0$ and
$\frac{\partial l}{\partial a} = \frac{1}{2}\sqrt{\frac{k}{a}} > 0$
so w increases and l decreases when a decreases.

[1]This Version of Workbook Chapter 7: September 29, 2004

Worksheet 7: Partial Differentiation

Quick Questions

(1) Obtain the first-order partial derivatives of each of the following functions:

$$f(x,y) = 4x^2y + 3xy^3 + 6x; \qquad g(x,y) = e^{2x+3y}; \qquad h(x,y) = \frac{x+y}{x-y}.$$

(2) Show that the function

$$f(x,y) = \sqrt{xy} + \frac{x^2}{y}$$

is homogeneous of degree 1. Find both partial derivatives and hence verify that it satisfies Euler's Theorem.

(3) A firm has production function $Q(K,L) = KL^2$, and faces demand function $P^d(Q) = 120 - 1.5Q$.
(a) Find the marginal products of labour and capital when $L = 3$ and $K = 4$.
(b) Write down the firm's revenue, R, as a function of output, Q, and find its marginal revenue.
(c) Hence, using the chain rule, find the marginal revenue product of labour when $L = 3$ and $K = 4$.

(4) A consumer has a quasi-linear utility function $u(x_1,x_2) = x_1^{\frac{1}{2}} + x_2$. Find the marginal utilities of both goods and the marginal rate of substitution. What does the MRS tell us about the shape of the indifference curves? Sketch the indifference curves.

(5) For a firm with Cobb-Douglas production function $Y(K,L) = aK^\alpha L^\beta$, show that the marginal rate of technical substitution depends on the capital-labour ratio. Show on a diagram what this means for the shape of the isoquants.

(6) The supply and demand functions in a market are

$$Q^s = kp^2 \quad \text{and} \quad Q^d = D(p)$$

where k is a positive constant and the demand function slopes down: $D'(p) < 0$.
(a) Write down the equation satisfied by the equilibrium price.
(b) Use implicit differentiation to find how the price changes if k increases.

Longer Questions

(1) A consumer's demand for good 1 depends on its own price, p_1, the price of good 2, p_2, and his income, y, according to the formula

$$x_1(p_1,p_2,y) = \frac{y}{p_1 + \sqrt{p_1p_2}}.$$

(a) Show that $x_1(p_1,p_2,y)$ is homogeneous of degree zero. What is the economic interpretation of this property?
(b) The own-price elasticity of demand is given by the formula

$$\epsilon_{11}(p_1,p_2,y) = \frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}.$$

Show that $|\epsilon_{11}|$ is always less than 1.

(2) Tony and Gordon have utility functions:

$$\begin{aligned} \text{Tony} &: \quad U_T(x,y) = x^4y^2 \\ \text{Gordon} &: \quad U_G(x,y) = \ln x + \tfrac{1}{2}\ln y \end{aligned}$$

(a) Find their marginal utilities for both goods, and show that they both have the same marginal rate of substitution.
(b) What is the equation of Tony's indifference curve corresponding to a utility level of 16? Show that it is convex, and draw it.
(c) Does Gordon have the same indifference curves? Explain carefully how their indifference curves are related.
(d) Show that Gordon has "diminishing marginal utility", but Tony does not.
(e) Do Tony and Gordon have the same preferences?

(3) Robinson Crusoe spends his 9 hour working day either fishing or looking for coconuts. If he spends t hours fishing, and s hours looking for coconuts, he will catch $f(t) = t^{\frac{1}{2}}$ fish, and collect $c(s) = 2s^{\frac{1}{3}}$ coconuts.

(a) If Crusoe's utility function is $U(f,c) = \ln(f) + \ln(c)$, show that the marginal rate at which he will substitute coconuts for fish is given by the expression:

$$MRS = -\frac{c}{f}$$

(b) Show that production possibility frontier of the economy is:

$$f^2 + \left(\frac{c}{2}\right)^3 = 9$$

(c) Find an expression for the marginal rate of transfomation between fish and coconuts.
(d) Crusoe plans to spend 1 hour fishing, and the rest of his time collecting coconuts. If he adopts this plan:
  (i) How many of each can he consume?
  (ii) What is his MRS?
  (iii) What is his MRT?
  (iv) Would you recommend that he should spend more or less time fishing? Why? Draw a diagram to illustrate.
(e) What would be the optimal allocation of his time?