CHAPTER 9

Constrained Optimisation

Rational economic agents are assumed to make choices that maximise their utility or profit. But their **choices are usually constrained** – for example the consumer's choice of consumption bundle is constrained by his income. In this chapter we look at methods for solving optimisation problems with constraints: in particular the **method of Lagrange multipliers**. We apply them to **consumer choice**, **cost minimisation**, and other economic problems.

1. Consumer Choice

Suppose there are two goods available, and a consumer has preferences represented by the utility function $u(x_1, x_2)$ where x_1 and x_2 are the amounts of the goods consumed. The prices of the goods are p_1 and p_2, and the consumer has a fixed income m. He wants to choose his consumption bundle (x_1, x_2) to maximise his utility, subject to the constraint that the total cost of the bundle does not exceed his income. Provided that the utility function is strictly increasing in x_1 and x_2, we know that he will want to use all his income.

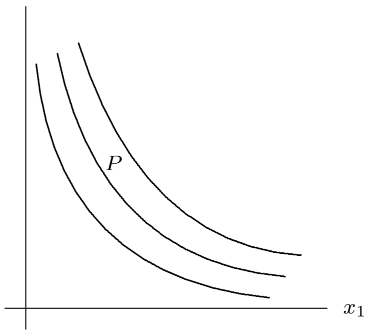
The consumer's optimisation problem is:

$$\max_{x_1,x_2} u(x_1, x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m$$

- The objective function is $u(x_1, x_2)$
- The choice variables are x_1 and x_2
- The constraint is $p_1x_1 + p_2x_2 = m$

There are three methods for solving this type of problem.

1.1. Method 1: Draw a Diagram and Think About the Economics



Provided the utility function is *well-behaved* (increasing in x_1 and x_2 with strictly convex indifference curves), the highest utility is obtained at the point P where the budget constraint is tangent to an indifference curve.

The slope of the budget constraint is: $-\dfrac{p_1}{p_2}$

and the slope of the indifference curve is: $MRS = -\dfrac{MU_1}{MU_2}$

(See Chapters 2 and 7 for these results.)

Hence we can find the point P:

(1) It is on the budget constraint:

$$p_1x_1 + p_2x_2 = m$$

(2) where the slope of the budget constraint equals the slope of the indifference curve:

$$\frac{p_1}{p_2} = \frac{MU_1}{MU_2}$$

In general, this gives us two equations that we can solve to find x_1 and x_2.

Examples 1.1: A consumer has utility function $u(x_1, x_2) = x_1x_2$. The prices of the two goods are $p_1 = 3$ and $p_2 = 2$, and her income is $m = 24$. How much of each good will she consume?

(i) The consumer's problem is:

$$\max_{x_1,x_2} x_1x_2 \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

The utility function $u(x_1, x_2) = x_1x_2$ is Cobb-Douglas (see Chapter 7). Hence it is well-behaved. The marginal utilities are:

$$MU_1 = \frac{\partial u}{\partial x_1} = x_2 \quad \text{and} \quad MU_2 = \frac{\partial u}{\partial x_2} = x_1$$

(ii) The optimal bundle:

(1) is on the budget constraint:

$$3x_1 + 2x_2 = 24$$

(2) where the budget constraint is tangent to an indifference curve:

$$\begin{aligned} \frac{p_1}{p_2} &= \frac{MU_1}{MU_2} \\ \Rightarrow \frac{3}{2} &= \frac{x_2}{x_1} \end{aligned}$$

(iii) From the tangency condition: $3x_1 = 2x_2$. Substituting this into the budget constraint:

$$\begin{aligned} 2x_2 + 2x_2 &= 24 \\ \Rightarrow x_2 &= 6 \\ \text{and hence } x_1 &= 4 \end{aligned}$$

The consumer's optimal choice is 4 units of good 1 and 6 units of good 2.

1.2. Method 2: Use the Constraint to Substitute for one of the Variables

If you did A-level maths, this may seem to be the obvious way to solve this type of problem. However it often gives messy equations and is rarely used in economic problems because it doesn't give much economic insight. Consider the example above:

$$\max_{x_1,x_2} x_1x_2 \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

From the constraint, $x_2 = 12 - \frac{3x_1}{2}$. By substituting this into the objective function, we can write the problem as:

$$\max_{x_1} \left(12x_1 - \frac{3x_1^2}{2}\right)$$

So, we have transformed it into an *unconstrained* optimisation problem in one variable. The first-order condition is:

$$12 - 3x_1 = 0 \quad \Rightarrow \quad x_1 = 4$$

Substituting back into the equation for x_2 we find, as before, that $x_2 = 6$. It can easily be checked that the second-order condition is satisfied.

1.3. The Most General Method: The Method of Lagrange Multipliers

This method is important because it can be used for a wide range of constrained optimisation problems. For the consumer problem:

$$\max_{x_1, x_2} u(x_1, x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m$$

(1) Write down the Lagrangian function:

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - m)$$

λ is a new variable, which we introduce to help solve the problem.
It is called a *Lagrange Multiplier.*

(2) Write down the first-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \end{aligned}$$

(3) Solve the first-order conditions to find x_1 and x_2.

Provided that the utility function is well-behaved (increasing in x_1 and x_2 with strictly convex indifference curves) then the values of x_1 and x_2 that you obtain by this procedure will be the optimum bundle.[1]

Examples 1.2: A consumer has utility function $u(x_1, x_2) = x_1x_2$. The prices of the two goods are $p_1 = 3$ and $p_2 = 2$, and her income is $m = 24$. Use the Lagrangian method to find how much of each good she will she consume.

(i) The consumer's problem is, as before:

$$\max_{x_1, x_2} x_1x_2 \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

Again, since the utility function is Cobb-Douglas, it is well-behaved.

(ii) The Lagrangian function is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1x_2 - \lambda(3x_1 + 2x_2 - 24)$$

[1] In this *Workbook* we will simply use the Lagrangian method, without explaining why it works. For some explanation, see *Anthony & Biggs* section 21.2.

(iii) First-order conditions:

$$\begin{array}{lcrcl}
\dfrac{\partial \mathcal{L}}{\partial x_1} = x_2 - 3\lambda & \Rightarrow & x_2 & = & 3\lambda \\
\dfrac{\partial \mathcal{L}}{\partial x_2} = x_1 - 2\lambda & \Rightarrow & x_1 & = & 2\lambda \\
\dfrac{\partial \mathcal{L}}{\partial \lambda} = -(3x_1 + 2x_2 - 24) & \Rightarrow & 3x_1 + 2x_2 & = & 24
\end{array}$$

(iv) Eliminating λ from the first two equations gives:

$$3x_1 = 2x_2$$

Substituting for x_1 in the third equation:

$$2x_2 + 2x_2 = 24 \quad \Rightarrow \quad x_2 = 6 \quad \text{and hence} \quad x_1 = 4$$

Thus we find (again) that the optimal bundle is 4 units of good 1 and 6 of good 2.

1.4. Some Useful Tricks

1.4.1. *Solving the Lagrangian first-order conditions.* In the example above, the first two equations of the first-order conditions are:

$$\begin{array}{rcl}
x_2 & = & 3\lambda \\
x_1 & = & 2\lambda
\end{array}$$

There are several (easy) ways to eliminate λ to obtain $3x_1 = 2x_2$. But one way, which is particularly useful in Lagrangian problems when the equations are more complicated, is to *divide* the two equations, so that λ cancels out:

$$\frac{x_2}{x_1} = \frac{3\lambda}{2\lambda} \quad \Rightarrow \quad \frac{x_2}{x_1} = \frac{3}{2}$$

Whenever you have two equations: $\begin{array}{rcl} A & = & B \\ C & = & D \end{array}$

where A, B, C and D are non-zero algebraic expressions, you can write: $\dfrac{A}{C} = \dfrac{B}{D}$.

1.4.2. *Transforming the objective function.* Remember from Chapter 7 that if preferences are represented by a utility function $u(x_1, x_2)$, and f is an increasing function, then $f(u)$ represents the same preference ordering. Sometimes we can use this to reduce the algebra needed to solve a problem. Suppose we want to solve:

$$\max_{x_1, x_2} x_1^{\frac{3}{4}} x_2^{\frac{1}{4}} \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

Preferences are represented by a Cobb-Douglas utility function $u = x_1^{\frac{3}{4}} x_2^{\frac{1}{4}}$, so they could equivalently be represented by:

$$\ln u = \tfrac{3}{4} \ln x_1 + \tfrac{1}{4} \ln x_2 \quad \text{or} \quad 4 \ln u = 3 \ln x_1 + \ln x_2$$

You can check that solving the problem:

$$\max_{x_1, x_2} 3 \ln x_1 + \ln x_2 \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

gives exactly the same answer as the original problem, but that the equations are simpler.

1.5. Well-Behaved Utility Functions

We have seen that for the optimisation methods described above to work, it is important that the utility function is *well-behaved*: increasing, with strictly convex indifference curves. Two examples of well-behaved utility functions are:

- The Cobb-Douglas Utility Function
$$u(x_1, x_2) = x_1^a x_2^b$$
- The Logarithmic Utility Function
$$u(x_1, x_2) = a \ln x_1 + b \ln x_2$$

(We know from Chapter 7 that the Cobb-Douglas function is well-behaved. The logarithmic function is a monotonic increasing transformation of the Cobb-Douglas, so it represents the same preferences and is also well-behaved.) Two other useful forms of utility function are:

- The CES Utility Function[2]
$$u(x_1, x_2) = \left(ax_1^{-\rho} + bx_2^{-\rho}\right)^{-\frac{1}{\rho}}$$
where a and b are positive parameters, and ρ is a parameter greater than -1.
- The Quasi-Linear Utility Function
$$u(x_1, x_2) = v(x_1) + x_2$$
where $v(x_1)$ is any increasing concave function.

Both of these are well-behaved. To prove it, you can show that:

- the function is increasing, by showing that the marginal utilities are positive
- the indifference curves are convex as we did for the Cobb-Douglas case (Chapter 7 section 4.2) by showing that the MRS gets less negative as you move along an indifference curve.

EXERCISES 9.1: **Consumer Choice**

(1) Use Method 1 to find the optimum consumption bundle for a consumer with utility function $u(x, y) = x^2 y$ and income $m = 30$, when the prices of the goods are $p_x = 4$ and $p_y = 5$. Check that you get the same answers by the Lagrangian method.

(2) A consumer has a weekly income of 26 (after paying for essentials), which she spends on restaurant Meals and Books. Her utility function is $u(M, B) = 3M^{\frac{1}{2}} + B$, and the prices are $p_M = 6$ and $p_B = 4$. Use the Lagrangian method to find her optimal consumption bundle.

(3) Use the trick of transforming the objective function (section 1.4.2) to solve:
$$\max_{x_1, x_2} x_1^{\frac{3}{4}} x_2^{\frac{1}{4}} \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

Further Reading and Exercises

- *Varian* "Intermediate Microeconomics", Chapter 5, covers the economic theory of Consumer Choice. The Appendix to Chapter 5 explains the same three methods for solving choice problems that we have used in this section.
- *Jacques* §5.5 and §5.6
- *Anthony & Biggs* §§21.2, 22.2, and 22.3

[2] "Constant Elasticity of Substitution" – this refers to another property of the function

2. Cost Minimisation

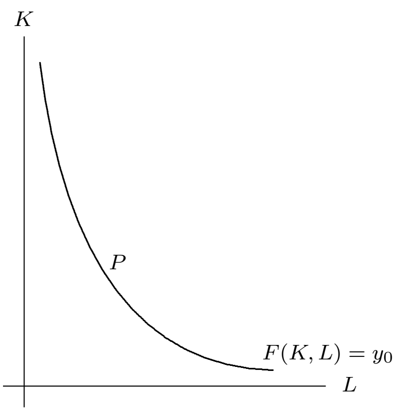
Consider a competitive firm with production function $F(K,L)$. Suppose that the wage rate is w, and the rental rate for capital is r. Suppose that the firm wants to produce a particular amount of output y_0 at minimum cost. How much labour and capital should it employ?

The firm's optimisation problem is:
$$\min_{K,L}(rK + wL) \quad \text{subject to} \quad F(K,L) = y_0$$

- The objective function is $rK + wL$
- The choice variables are K and L
- The constraint is $F(K,L) = y_0$

We can use the same three methods here as for the consumer choice problem, but we will ignore method 2 because it is generally less useful.

2.1. Method 1: Draw a Diagram and Think About the Economics



Draw the isoquant of the production function representing combinations of K and L that can be used to produce output y_0. (See Chapter 7.)

The slope of the isoquant is: $\quad MRTS = -\dfrac{MPL}{MPK}$

Draw the isocost lines where $rK + wL = constant$.

The slope of the isocost lines is: $\quad -\dfrac{w}{r}$

Provided the isoquant is convex, the lowest cost is achieved at the point P where an isocost line is tangent to the isoquant.

Hence we can find the point P:

(1) It is on the isoquant:
$$F(K,L) = y_0$$

(2) where the slope of the isocost lines equals the slope of the isoquant:
$$\frac{w}{r} = \frac{MPL}{MPK}$$

This gives us two equations that we can solve to find K and L.

Examples 2.1: If the production function is $F(K,L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$, the wage rate is 5, and the rental rate of capital is 20, what is the minimum cost of producing 40 units of output?

(i) The problem is:
$$\min_{K,L}(20K + 5L) \quad \text{subject to} \quad K^{\frac{1}{3}}L^{\frac{2}{3}} = 40$$

The production function $F(K,L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$ is Cobb-Douglas, so the isoquant is convex. The marginal products are:

$$MPK = \frac{\partial F}{\partial K} = \tfrac{1}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} \quad \text{and} \quad MPL = \frac{\partial F}{\partial L} = \tfrac{2}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}$$

(ii) The optimal choice of inputs:

(1) is on the isoquant:

$$K^{\frac{1}{3}}L^{\frac{2}{3}} = 40$$

(2) where an isocost line is tangent to it:

$$\begin{aligned} \frac{w}{r} &= \frac{MPL}{MPK} \\ \Rightarrow \frac{5}{20} &= \frac{\frac{2}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}}{\frac{1}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}} \\ \Rightarrow \frac{1}{8} &= \frac{K}{L} \end{aligned}$$

(iii) From the tangency condition: $L = 8K$. Substituting this into the isoquant:

$$\begin{aligned} K^{\frac{1}{3}}(8K)^{\frac{2}{3}} &= 40 \\ \Rightarrow K &= 10 \\ \text{and hence } L &= 80 \end{aligned}$$

(iv) With this optimal choice of K and L, the cost is: $20K + 5L = 20 \times 10 + 5 \times 80 = 600$.

2.2. The Method of Lagrange Multipliers

For the cost minimisation problem, the method is analagous to the one for consumer choice:

$$\min_{K,L}(rK + wL) \quad \text{subject to} \quad F(K,L) = y_0$$

(1) Write down the Lagrangian function:

$$\mathcal{L}(K,L,\lambda) = rK + wL - \lambda(F(K,L) - y_0)$$

(2) Write down the first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial K} = 0, \quad \frac{\partial \mathcal{L}}{\partial L} = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

(3) Solve the first-order conditions to find K and L.

Provided that the isoquants are convex, this procedure obtains the optimal values of K and L.

Examples 2.2: If the production function is $F(K,L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$, the wage rate is 5, and the rental rate of capital is 20, use the Lagrangian method to find the minimum cost of producing 40 units of output.

(i) The problem is, as before:

$$\min_{K,L}(20K + 5L) \quad \text{subject to} \quad K^{\frac{1}{3}}L^{\frac{2}{3}} = 40$$

and since the production function is Cobb-Douglas, the isoquant is convex.

(ii) The Lagrangian function is:

$$\mathcal{L}(K,L,\lambda) = 20K + 5L - \lambda(K^{\frac{1}{3}}L^{\frac{2}{3}} - 40)$$

(iii) First-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K} &= 20 - \tfrac{1}{3}\lambda K^{-\frac{2}{3}} L^{\frac{2}{3}} &\Rightarrow\quad & 60 &= \lambda K^{-\frac{2}{3}} L^{\frac{2}{3}} \\ \frac{\partial \mathcal{L}}{\partial L} &= 5 - \tfrac{2}{3}\lambda K^{\frac{1}{3}} L^{-\frac{1}{3}} &\Rightarrow\quad & 15 &= 2\lambda K^{\frac{1}{3}} L^{-\frac{1}{3}} \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= -(K^{\frac{1}{3}} L^{\frac{2}{3}} - 40) &\Rightarrow\quad & K^{\frac{1}{3}} L^{\frac{2}{3}} &= 40 \end{aligned}$$

(iv) Dividing the first two equations to eliminate λ:

$$\frac{60}{15} = \frac{K^{-\frac{2}{3}} L^{\frac{2}{3}}}{2K^{\frac{1}{3}} L^{-\frac{1}{3}}} \quad \text{which simplifies to} \quad L = 8K$$

Substituting for L in the third equation:

$$K^{\frac{1}{3}}(8K)^{\frac{2}{3}} = 40 \quad \Rightarrow \quad K = 10 \quad \text{and hence} \quad L = 80$$

(v) Thus, as before, the optimal choice is 10 units of capital and 80 of labour, which means that the cost is 600.

EXERCISES 9.2: **Cost Minimisation**

(1) A firm has production function $F(K, L) = 5K^{0.4}L$. The wage rate is $w = 10$ and the rental rate of capital is $r = 12$. Use Method 1 to determine how much labour and capital the firm should employ if it wants to produce 300 units of output. What is the total cost of doing so?
Check that you get the same answer by the Lagrangian method.

(2) A firm has production function $F(K, L) = 30(K^{-1} + L^{-1})^{-1}$. Use the Lagrangian method to find how much labour and capital it should employ to produce 70 units of output, if the wage rate is 8 and the rental rate of capital is 2.
Note: the production function is CES, so has convex isoquants.

Further Reading and Exercises

- *Jacques* §5.5 and §5.6
- *Anthony & Biggs* §§21.1, 21.2 and 21.3
- *Varian* "Intermediate Microeconomics", Chapter 20, covers the economics of Cost Minimisation.

3. The Method of Lagrange Multipliers

To try to solve any problem of the form:

$$\max_{x_1,x_2} F(x_1,x_2) \quad \text{subject to} \quad g(x_1,x_2)=c$$

or:

$$\min_{x_1,x_2} F(x_1,x_2) \quad \text{subject to} \quad g(x_1,x_2)=c$$

you can write down the Lagrangian:

$$\mathcal{L}(x_1,x_2,\lambda)=F(x_1,x_2)-\lambda(g(x_1,x_2)-c)$$

and look for a solution of the three first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1}=0, \quad \frac{\partial \mathcal{L}}{\partial x_2}=0, \quad \frac{\partial \mathcal{L}}{\partial \lambda}=0$$

Note that the condition $\frac{\partial \mathcal{L}}{\partial \lambda}=0$ always gives you the constraint.

3.1. Max or Min?

But if you find a solution to the Lagrangian first-order conditions, how do you know whether it is a maximum or a minimum? And what if there are several solutions? The answer is that in general you need to look at the second-order conditions, but unfortunately the SOCs for constrained optimisation problems are complicated to write down, so will not be covered in this *Workbook*.

However, when we apply the method to economic problems, we can often manage without second-order conditions. Instead, we think about the *shapes* of the objective function and the constraint, and find that either it is the type of problem in which the objective function can *only* have a maximum, or that it is a problem that can only have a minimum. In such cases, the method of Lagrange multipliers will give the required solution.

If you look back at the examples in the previous two sections, you can see that the Lagrangian method gives you just the same equations as you get when you "draw a diagram and think about the economics" – what the method does is to find tangency points.

But when you have a problem in which either the objective function or constraint doesn't have the standard economic properties of convex indifference curves or isoquants you cannot rely on the Lagrangian method, because a tangency point that it finds may not be the required maximum or minimum.

EXAMPLES 3.1: *Standard and non-standard problems*

(i) If the utility function is well-behaved (increasing, with convex indifference curves) the problem:

$$\max_{x_1,x_2} u(x_1,x_2) \quad \text{subject to} \quad p_1x_1+p_2x_2=m$$

can be solved by the Lagrangian method.

(ii) If the production function has convex isoquants the problem:

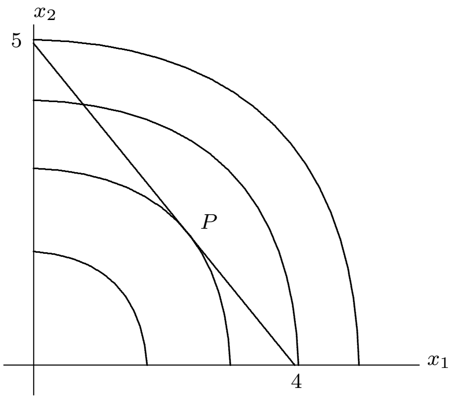
$$\min_{K,L}(rK+wL) \quad \text{subject to} \quad F(K,L)=y_0$$

can be solved by the Lagrangian method.

(iii) Consider a consumer choice problem with *concave* indifference curves:

$$\max_{x_1,x_2} u(x_1, x_2) \quad \text{subject to} \quad 5x_1 + 4x_2 = 20$$

where $u(x_1, x_2) = x_1^2 + x_2^2$.



The tangency point P is the point on the budget constraint where utility is *minimised.*

Assuming that the amounts x_1 and x_2 cannot be negative, the point on the budget constraint where utility is maximised is $x_1 = 0$, $x_2 = 5$.

If we write the Lagrangian:
$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + x_2^2 - \lambda(5x_1 + 4x_2 - 20)$
the first-order conditions are:

$$2x_1 = 5\lambda, \quad 2x_2 = 4\lambda, \quad 5x_1 + 4x_2 = 20$$

Solving, we obtain $x_1 = 2.44$, $x_2 = 1.95$. The Lagrangian method has found the tangency point P, which is where utility is minimised.

3.2. The Interpretation of the Lagrange Multiplier

In the examples of the Lagrangian method so far, we did not bother to calculate the value of λ satisfying the first-order conditions – we simply eliminated it and solved for the choice variables. However, this value does have a meaning.

> In an optimisation problem with objective function $F(x, y)$ and constraint $g(x, y) = c$, let F^* be the value of the objective function at the optimum. The value of the Lagrange multipler λ indicates how much F^* would increase if there were a small increase in c:
>
> $$\lambda = \frac{dF^*}{dc}$$

This means, for example, that for the cost minimisation problem:

$$\min_{K,L}(rK + wL) \quad \text{subject to} \quad F(K, L) = y$$

the Lagrange multiplier tells us how much the cost would increase if there were a small increase in the amount of output to be produced – that is, the marginal cost of output.

We will prove this result using differentials (see Chapter 7). The Lagrangian is:

$$\mathcal{L} = rK + wL - \lambda(F(K, L) - y)$$

and the first-order conditions are:

$$\begin{aligned} r &= \lambda F_K \\ w &= \lambda F_L \\ F(K, L) &= y \end{aligned}$$

These equations can be solved to find the optimum values of capital K^* and labour L^*, and the Lagrange multiplier λ^*. The cost is then $C^* = rK^* + wL^*$.

K^*, L^* and C^* all depend on the the level of output to be produced, y. Suppose there is small change in this amount, dy. This will lead to small changes in the optimal factor choices and the cost, dK^*, dL^*, and dC^*. We can work out how big the change in cost will be:

First, taking the differential of the constraint $F(K^*, L^*) = y$:

$$dy = F_K dK^* + F_L dL^*$$

then, taking the differential of the cost $C^* = rK^* + wL^*$ and using the first-order conditions:

$$\begin{aligned} dC^* &= r dK^* + w dL^* \\ &= \lambda^* F_K dK^* + \lambda^* F_L dL^* \\ &= \lambda^* dy \\ \Rightarrow \frac{dC^*}{dy} &= \lambda^* \end{aligned}$$

So the value of the Lagrange multiplier tells us the marginal cost of output.

Similarly for utility maximisation:

$$\max_{x_1,x_2} u(x_1, x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m$$

the Lagrange multiplier gives the value of $\frac{du^*}{dm}$, which is the marginal utility of income.

This, and the general result in the box above, can be proved in the same way.

Examples 3.2: *The Lagrange Multiplier*

In Examples 2.2, for the production function $F(K, L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$, with wage rate 5 and rental rate of capital 20, we found the minimum cost of producing 40 units of output. The Lagrangian and first-order conditions were:

$$\mathcal{L}(K, L, \lambda) = 20K + 5L - \lambda(K^{\frac{1}{3}}L^{\frac{2}{3}} - 40)$$

$$\begin{aligned} 60 &= \lambda K^{-\frac{2}{3}}L^{\frac{2}{3}} \\ 15 &= 2\lambda K^{\frac{1}{3}}L^{-\frac{1}{3}} \\ K^{\frac{1}{3}}L^{\frac{2}{3}} &= 40 \end{aligned}$$

Solving these we found the optimal choice $K = 10$, $L = 80$, with total cost 600. Substituting these values of K and L back into the first first-order condition:

$$\begin{aligned} 60 &= \lambda 10^{-\frac{2}{3}}80^{\frac{2}{3}} \\ \Rightarrow \lambda &= 15 \end{aligned}$$

Hence the firm's marginal cost is 15. So we can say that producing 41 units of output would cost approximately 615. (In fact, in this example, marginal cost is constant: the value of λ does not depend on the number of units of output.)

3.3. Problems with More Variables and Constraints

The Lagrangian method can be generalised in an obvious way to solve problems in which there are more variables and several constraints. For example, to solve the problem:

$$\max_{x_1,x_2,x_3,x_4} F(x_1,x_2,x_3,x_4) \quad \text{subject to} \quad g_1(x_1,x_2,x_3,x_4) = c_1 \text{ and } g_2(x_1,x_2,x_3,x_4) = c_2$$

the Lagrangian would be:

$$\mathcal{L}(x_1,x_2,x_3,x_4,\lambda_1,\lambda_2) = F(x_1,x_2,x_3,x_4) - \lambda_1(g_1(x_1,x_2,x_3,x_4) - c_1) - \lambda_2(g_2(x_1,x_2,x_3,x_4) - c_2)$$

giving six first-order conditions to solve for the optimal choice.

Exercises 9.3: **The Method of Lagrange Multipliers**

(1) Find the Lagrange Multiplier and hence the marginal cost of the firm in Exercises 9.2, Question 2.

(2) Find the optimum consumption bundle for a consumer with utility $u = x_1x_2x_3$ and income 36, when the prices of the goods are $p_1 = 1$, $p_2 = 6$, $p_3 = 10$.

(3) A consumer with utility function $u(x_1, x_2)$ has income $m = 12$, and the prices of the goods are $p_1 = 3$ and $p_2 = 2$. For each of the following cases, decide whether the utility function is well-behaved, and determine the optimal choices: (a) $u = x_1 + x_2$ (b) $u = 3x_1^{2/3} + x_2$ (c) $u = \min(x_1, x_2)$

(4) A firm has production function $F(K, L) = \frac{1}{4}\left(K^{1/2} + L^{1/2}\right)$. The wage rate is $w = 1$ and the rental rate of capital is $r = 3$.

   (a) How much capital and labour should the firm employ to produce y units of output?

   (b) Hence find the cost of producing y units of output (the firm's cost function).

   (c) Differentiate the cost function to find the marginal cost, and verify that it is equal to the value of the Lagrange multiplier.

(5) By following similar steps to those we used for the cost minimisation problem in section 3.2 prove that for the utility maximisation problem:

$$\max_{x_1,x_2} u(x_1,x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m$$

the Lagrange multiplier is equal to the marginal utility of income.

Further Reading and Exercises

- *Anthony & Biggs* Chapter 22 gives a general formulation of the Lagrangian method, going beyond what we have covered here.

4. Some More Examples of Constrained Optimisation Problems in Economics

4.1. Production Possibilities

Robinson Crusoe spends his 8 hour working day either fishing or looking for coconuts. If he spends t hours fishing and s hours looking for coconuts he will catch $f(t) = t^{\frac{1}{2}}$ fish and find $c(s) = 3s^{\frac{1}{2}}$ coconuts. Crusoe's utility function is $u = \ln f + \ln c$.

The optimal pattern of production in this economy is the point on the production possibility frontier (ppf) where Crusoe's utility is maximised.

The time taken to catch f fish is $t = f^2$, and the time take to find c coconuts is $s = \left(\frac{c}{3}\right)^2$. Since he has 8 hours in total, the ppf is given by:

$$f^2 + \left(\frac{c}{3}\right)^2 = 8$$

To find the optimal point on the ppf we have to solve the problem:

$$\max_{c,f}(\ln c + \ln f) \quad \text{subject to} \quad f^2 + \left(\frac{c}{3}\right)^2 = 8$$

Note that the ppf is concave, and the utility function is well-behaved (it is the log of a Cobb-Douglas). Hence the optimum is a tangency point, which we could find either by the Lagrangian method, or using the condition that the marginal rate of substitution must be equal to the marginal rate of transformation – see the similar problem on Worksheet 7.

4.2. Consumption and Saving

A consumer lives for two periods (work and retirement). His income is 100 in the first period, and zero in the second. The interest rate is 5%. His lifetime utility is given by:

$$U(c_1, c_2) = c_1^{\frac{1}{2}} + 0.9c_2^{\frac{1}{2}}$$

If he consumes c_1 in the first period he will save $(100 - c_1)$. So when he is retired he can consume:

$$c_2 = 1.05(100 - c_1)$$

This is his lifetime budget constraint. Rearranging, it can be written:

$$c_1 + \frac{c_2}{1.05} = 100$$

Hence the consumer's optimisation problem is:

$$\max_{c_1,c_2}\left(c_1^{\frac{1}{2}} + 0.9c_2^{\frac{1}{2}}\right) \quad \text{subject to} \quad c_1 + \frac{c_2}{1.05} = 100$$

which can be solved using the Lagrangian:

$$\mathcal{L}(c_1, c_2, \lambda) = c_1^{\frac{1}{2}} + 0.9c_2^{\frac{1}{2}} - \lambda\left(c_1 + \frac{c_2}{1.05} - 100\right)$$

to determine the optimal consumption and saving plan for the consumer's lifetime.

4.3. Labour Supply

A consumer has utility

$$U(C, R) = 3\ln C + \ln R$$

where C is her amount of consumption, and R is the number of hours of leisure (relaxation) she takes each day. The hourly wage rate is $w = 4$ (measured in units of consumption). She has a non-labour income $m = 8$ (consumption units). She needs 10 hours per day for eating and sleeping; in the remainder she can work or take leisure.

Suppose that we want to know her labour supply – how many hours she chooses to work. First we need to know the budget constraint. If she takes R units of leisure, she will work for $14 - R$ hours, and hence earn $4(14 - R)$. Then she will be able to consume:

$$C = 4(14 - R) + 8$$

units of consumption. This is the budget constraint. Rearranging, we can write it as:

$$C + 4R = 64$$

So we need to solve the problem:

$$\max_{C,R} (3\ln C + \ln R) \quad \text{subject to} \quad C + 4R = 64$$

to find the optimal choice of leisure R, and hence the number of hours of work.

Exercises 9.4: **More Constrained Optimisation Problems**
Complete the solution of each of three problems discussed in this section.

5. Determining Demand Functions

5.1. Consumer Demand

In the consumer choice problems in section 1 we determined the optimal consumption bundle, given the utility function and *particular values* for the prices of the goods, and income. If we solve the same problem for *general values* p_1, p_2, and m, we can determine the consumer's demands for the goods as a function of prices and income:

$$x_1 = x_1(p_1, p_2, m) \quad \text{and} \quad x_2 = x_2(p_1, p_2, m)$$

EXAMPLES 5.1: A consumer with a fixed income m has utility function $u(x_1, x_2) = x_1^2 x_2^3$. If the prices of the goods are p_1 and p_2, find the consumer's demand functions for the two goods.

The optimisation problem is:

$$\max_{x_1, x_2} x_1^2 x_2^3 \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 = m$$

The Lagrangian is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 x_2^3 - \lambda (p_1 x_1 + p_2 x_2 - m)$$

Differentiating to obtain the first-order conditions:

$$\begin{aligned} 2x_1 x_2^3 &= \lambda p_1 \\ 3x_1^2 x_2^2 &= \lambda p_2 \\ p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

Dividing the first two equations:

$$\frac{2x_2}{3x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = \frac{3p_1 x_1}{2p_2}$$

Substituting into the budget constraint we find:

$$\begin{aligned} p_1 x_1 + \frac{3p_1 x_1}{2} &= m \\ \Rightarrow \quad x_1 &= \frac{2m}{5p_1} \\ \text{and hence} \quad x_2 &= \frac{3m}{5p_2} \end{aligned}$$

These are the consumer's demand functions. Demand for each good is an increasing function of income and a decreasing function of the price of the good.

5.2. Cobb-Douglas Utility

Note that in Examples 5.1, the consumer spends $\frac{2}{5}$ of his income on good 1 and $\frac{3}{5}$ on good 2:

$$p_1 x_1 = \frac{2}{5} m \quad \text{and} \quad p_2 x_2 = \frac{3}{5} m$$

This is an example of a general result for Cobb-Douglas Utility functions:

A consumer with Cobb-Douglas utility: $u(x_1, x_2) = x_1^a x_2^b$
spends a constant fraction of income on each good:
$p_1 x_1 = \frac{a}{a+b} m$ and $p_2 x_2 = \frac{b}{a+b} m$

You could prove this by re-doing example 5.1 using the more general utility function $u(x_1, x_2) = x_1^a x_2^b$.

This result means that with Cobb-Douglas utility a consumer's demand for one good does not depend on the price of the other good.

5.3. Factor Demands

Similarly, if we solve the firm's cost minimisation problem:

$$\min_{K,L}(rK + wL) \quad \text{subject to} \quad F(K, L) = y_0$$

for general values of the factor prices r and w, and the output y_0, we can find the firm's *Conditional Factor Demands* (see Varian Chapter 20) – that is, its demand for each factor as a function of output and factor prices:

$$K = K(y_0, r, w) \quad \text{and} \quad L = L(y_0, r, w)$$

From these we can obtain the firm's cost function:

$$C(y, r, w) = rK(y, r, w) + wL(y, r, w)$$

EXERCISES 9.5: **Determining Demand Functions**

(1) A consumer has an income of y, which she spends on Meals and Books. Her utility function is $u(M, B) = 3M^{\frac{1}{2}} + B$, and the prices are p_M and p_B. Use the Lagrangian method to find her demand functions for the two goods.

(2) Find the conditional factor demand functions for a firm with production function $F(K, L) = KL$. If the wage rate and the rental rate for capital are both equal to 4, what is the firm's cost function $C(y)$?

(3) Prove that a consumer with Cobb-Douglas utility spends a constant fraction of income on each good.

Further Reading and Exercises

- *Anthony & Biggs* §§21.3 and 22.4
- *Varian* "Intermediate Microeconomics", Chapters 5 and 6 (including Appendices) for consumer demand, and Chapter 20 (including Appendix) for conditional factor demands and cost functions.

Solutions to Exercises in Chapter 9

EXERCISES 9.1:

(1) $MU_x = 2xy$, $MU_y = x^2$
Tangency Condition: $\frac{2xy}{x^2} = \frac{4}{5} \Rightarrow y = \frac{2}{5}x$
Budget Constraint: $4x + 5y = 30$
$\Rightarrow 6x = 30 \Rightarrow x = 5$, $y = 2$.

(2) $\max_{M,B} \left(3M^{\frac{1}{2}} + B\right)$ s.t. $6M + 4B = 26$
$\mathcal{L} = \left(3M^{\frac{1}{2}} + B\right) - \lambda(6M + 4B - 26)$
$\Rightarrow \frac{3}{2}M^{-\frac{1}{2}} = 6\lambda$, $1 = 4\lambda$, $6M + 4B = 26$
Eliminate $\lambda \Rightarrow M = 1$
Then b.c. $\Rightarrow 6 + 4B = 26 \Rightarrow B = 5$.

(3) $\mathcal{L} = (3 \ln x_1 + \ln x_2) - \lambda(3x_1 + 2x_2 - 24)$
$\Rightarrow \frac{3}{x_1} = 3\lambda$, $\frac{1}{x_2} = 2\lambda$, $3x_1 + 2x_2 = 24$
Solving $\Rightarrow x_2 = 3$, $x_1 = 6$.

EXERCISES 9.2:

(1) $\min_{K,L}(12K + 10L)$ s.t. $5K^{0.4}L = 300$
$MPL = 5K^{0.4}$, $MPK = 2K^{-0.6}L$
Tangency: $\dfrac{10}{12} = \dfrac{5K^{0.4}}{2K^{-0.6}L} \Rightarrow L = 3K$
Isoquant: $5K^{0.4}L = 300 \Rightarrow 15K^{1.4} = 300$
$\Rightarrow K = (20)^{\frac{1}{1.4}} = 8.50$, $L = 25.50$
Cost$=12K + 10L = 357$

(2) $\min_{K,L}(8L + 2K)$ s.t. $\dfrac{30}{L^{-1} + K^{-1}} = 70$
$\mathcal{L} = (8L + 2K) - \lambda\left(\frac{3}{L^{-1}+K^{-1}} - 7\right) \Rightarrow$
$8 = \frac{3\lambda}{L^2(L^{-1}+K^{-1})^2}$, $2 = \frac{3\lambda}{K^2(L^{-1}+K^{-1})^2}$
Eliminate $\lambda \Rightarrow K = 2L$. Substitute in:
$\frac{3}{L^{-1}+K^{-1}} = 7 \Rightarrow L = 3.5$, $K = 7$

EXERCISES 9.3:

(1) From 2^{nd} FOC in previous question:
$3\lambda = 2K^2(L^{-1} + K^{-1})^2$.
Substituting $L = 3.5$, $K = 7 \Rightarrow \lambda = 0.6$

(2) $\max_{x_1,x_2,x_3} x_1x_2x_3$ s.t. $x_1 + 6x_2 + 10x_3 = 36$
$\mathcal{L} = x_1x_2x_3 - \lambda(x_1 + 6x_2 + 10x_3 - 36)$
$\Rightarrow$ (i) $x_2x_3 = \lambda$, (ii) $x_1x_3 = 6\lambda$,
(iii) $x_1x_2 = 10\lambda$, (iv) $x_1+6x_2+10x_3 = 36$
Solving: (i) and (iii) $\Rightarrow x_1 = 10x_3$, and (ii) and (iii) $\Rightarrow x_2 = \frac{5}{3}x_3$.
Substituting in (iv): $30x_3 = 36$
$\Rightarrow x_3 = 1.2$, $x_1 = 12$ and $x_2 = 2$.

(3) (a) No. Perfect substitutes: u is linear, not strictly convex. Consumer buys good 2 only as it is cheaper: $x_1 = 0$, $x_2 = 6$.
(b) Yes. Quasi-linear; $x_1^{2/3}$ is concave.
$\mathcal{L} = (6x_1^{2/3} + x_2) - \lambda(3x_1 + 2x_2 - 12)$
$\Rightarrow 2x_1^{-1/3} = 3\lambda$, $1 = 2\lambda$,
and $3x_1 + 2x_2 = 12$
Eliminating $\lambda \Rightarrow 2x_1^{-1/3} = \frac{3}{2}$
$\Rightarrow x_1^{1/3} = \frac{4}{3} \quad \Rightarrow x_1 = 2.37$
and from budget constraint:
$7.11 + 2x_2 = 12 \quad \Rightarrow x_2 = 2.45$
(c) No. Perfect complements.
Optimum is on the budget constraint where $x_1 = x_2$:
$3x_1 + 2x_2 = 12$ and $x_1 = x_2$
$\Rightarrow x_1 = x_2 = 2.4$

(4) (a) $\mathcal{L} = 3K+L-\lambda\left(\frac{1}{4}\left(L^{1/2} + K^{1/2}\right) - y\right) \Rightarrow$
$24 = \lambda K^{-1/2}, 8 = \lambda L^{-1/2}$ and
$L^{1/2} + K^{1/2} = 4y$. Solving:
$K = y^2$, $L = 9y^2$.
(b) $C = 3K + L = 12y^2$
(c) $C' = 24y$.
From f.o.c. $\lambda = 24K^{1/2} = 24y$

(5) $\mathcal{L} = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - m)$
f.o.c.s: $u_1{=}\lambda p_1$, $u_2{=}\lambda p_2$, $p_1x_1 + p_2x_2{=}m$
Constraint: $dm = p_1dx_1 + p_2dx_2$
Utility: $du = u_1dx_1 + u_2dx_2 = \lambda p_1dx_1 + \lambda p_2dx_2 = \lambda dm \Rightarrow \frac{du}{dm} = \lambda$

EXERCISES 9.4:

(1) $\mathcal{L} = (\ln c + \ln f) - \lambda\left(f^2 + \left(\dfrac{c}{3}\right)^2 - 8\right)$
$\Rightarrow \dfrac{1}{f} = 2\lambda f$, $\dfrac{1}{c} = \dfrac{2\lambda c}{9}$, $f^2 + \left(\frac{c}{3}\right)^2 = 8$
Solving $\Rightarrow f = 2$, $c = 6$.

(2) $0.5c_1^{-\frac{1}{2}}{=}\lambda$, $0.45c_2^{-\frac{1}{2}}{=}\frac{\lambda}{1.05}$, $c_1 + \frac{c_2}{1.05} = 100$
Elim. $\lambda \Rightarrow c_2{=}.893c_1$ so $c_1 + \frac{.893}{1.05}c_1{=}100$
$\Rightarrow c_1 = 54.04$, $c_2 = 48.26$

(3) $\frac{3}{C} = \lambda$, $\frac{1}{R} = 4\lambda$, $C + 4R = 64$. Solving:
$C = 12R$, $\Rightarrow R = 4$, $C = 48$, 10 hrs work.

EXERCISES 9.5:

(1) $\mathcal{L} = (3M^{1/2} + B) - \lambda(p_MM + p_BB - y) \Rightarrow$
$\frac{3}{2}M^{-\frac{1}{2}} = \lambda p_M$, $1 = \lambda p_B$, $p_MM + p_BB = y$
Solving for the demand functions:
$M = 2.25p_B^2/p_M^2$, $B = y/p_B - 2.25p_B/p_M$

(2) $\mathcal{L} = (rK + wL) - \lambda(KL - y) \Rightarrow w = \lambda K$, $r = \lambda L$ and $KL = y$. Solving: $L = \sqrt{\frac{ry}{w}}$, $K = \sqrt{\frac{wy}{r}}$, $C(y, 4, 4) = 8\sqrt{y}$

(3) $\mathcal{L} = x_1^a x_2^b - \lambda(p_1 x_1 + p_2 x_2 - m)$ $\Rightarrow ax_1^{a-1} x_2^b = \lambda p_1$, $bx_1^a x_2^{b-1} = \lambda p_2$ and $p_1 x_1 + p_2 x_2 = m$. Elim. $\lambda \Rightarrow \frac{ax_2}{bx_1} = \frac{p_1}{p_2}$. Subst. in b.c.: $p_1 x_1 + \frac{b}{a} p_1 x_1 = m$ $\Rightarrow p_1 x_1 = \frac{a}{a+b} m$, $p_2 x_2 = \frac{b}{a+b} m$

[3]This Version of Workbook Chapter 9: September 15, 2003

Worksheet 9: Constrained Optimisation Problems

Quick Questions

(1) A consumer has utility function $u(x_1, x_2) = 2\ln x_1 + 3\ln x_2$, and income $m = 50$. The prices of the two goods are $p_1 = p_2 = 1$. Use the MRS condition to determine his consumption of the two goods. How will consumption change if the price of good 1 doubles? Comment on this result.

(2) Repeat the first part of question 1 using the Lagrangian method and hence determine the marginal utility of income.

(3) Is the utility function $u(x_1, x_2) = x_2 + 3x_1^2$ well-behaved? Explain your answer.

(4) A firm has production function $F(K, L) = 8KL$. The wage rate is 2 and the rental rate of capital is 1. The firm wants to produce output y.
  (a) What is the firm's cost minimisation problem?
  (b) Use the Lagrangian method to calculate its demands for labour and capital, in terms of output, y.
  (c) Evaluate the Lagrange multiplier and hence determine the firm's marginal cost.
  (d) What is the firm's cost function $C(y)$?
  (e) Check that you obtain the same expression for marginal cost by differentiating the cost function.

Longer Questions

(1) A rich student, addicted to video games, has a utility function given by $U = S^{\frac{1}{2}}N^{\frac{1}{2}}$, where S is the number of Sega brand games he owns and N is the number of Nintendo brand games he possesses (he owns machines that will allow him to play games of either brand). Sega games cost £16 each and Nintendo games cost £36 each. The student has disposable income of £2,880 after he has paid his battels, and no other interests in life.
  (a) What is his utility level, assuming he is rational?
  (b) Sega, realizing that their games are underpriced compared to Nintendo, raise the price of their games to £36 as well. By how much must the student's father raise his son's allowance to maintain his utility at the original level?
  (c) Comment on your answer to (b).

(2) George is a graduate student and he divides his working week between working on his research project and teaching classes in mathematics for economists. He estimates that his utility function for earning £W by teaching classes and spending R hours on his research is:
$$u(W, R) = W^{\frac{3}{4}}R^{\frac{1}{4}}.$$
He is paid £16 per hour for teaching and works for a total of 40 hours each week. How should he divide his time between teaching and research in order to maximize his utility?

(3) Maggie likes to consume goods and to take leisure time each day. Her utility function is given by $U = \frac{CH}{C+H}$ where C is the quantity of goods consumed per day and H is

the number of hours spent at leisure each day. In order to finance her consumption bundle Maggie works $24 - H$ hours per day. The price of consumer goods is £1 and the wage rate is £9 per hour.

(a) By showing that it is a CES function, or otherwise, check that the utility function is well-behaved.
(b) Using the Lagrangian method, find how many hours Maggie will work per day.
(c) The government decides to impose an income tax at a rate of 50% on all income. How many hours will Maggie work now? What is her utility level? How much tax does she pay per day?
(d) An economist advises the government that instead of setting an income tax it would be better to charge Maggie a lump-sum tax equal to the payment she would make if she were subject to the 50% income tax. How many hours will Maggie work now, when the income tax is replaced by a lump-sum tax yielding an equal amount? Compare Maggie's utility under the lump-sum tax regime with that under the income tax regime.

(4) A consumer purchases two good in quantities x_1 and x_2; the prices of the goods are p_1 and p_2 respectively. The consumer has a total income I available to spend on the two goods. Suppose that the consumer's preferences are represented by the utility function
$$u(x_1, x_2) = x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}}.$$
(a) Calculate the consumer's demands for the two goods.
(b) Find the own-price elasticity of demand for good 1, $\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}$. Show that if $p_1 = p_2$, then this elasticity is $-\frac{5}{4}$.
(c) Find the cross-price elasticity $\frac{p_2}{x_1}\frac{\partial x_1}{\partial p_2}$ when $p_1 = p_2$.

(5) There are two individuals, A and B, in an economy. Each derives utility from his consumption, C, and the fraction of his time spent on leisure, l, according to the utility function:
$$U = \ln(C) + \ln(l)$$
However, A is made very unhappy if B's consumption falls below 1 unit, and he makes a transfer, G, to ensure that it does not. B has no concern for A. A faces a wage rate of 10 per period, and B a wage rate of 1 per period.
(a) For what fraction of the time does each work, and how large is the transfer G?
(b) Suppose A is able to insist that B does not reduce his labour supply when he receives the transfer. How large should it be then, and how long should A work?