CHAPTER 10

Integration

Integration can be thought of as **the opposite of differentiation** but is also a method for **finding the area under a graph**. It is an important mathematical technique, which will be familiar if you have done A-level maths. In this chapter we look at techniques for integrating standard functions, including **integration by substitution** and **by parts** and at economic applications including calculation of **consumer surplus**.

1. The Reverse of Differentiation

If we have a function $y(x)$, we know how to find its *derivative* $\frac{dy}{dx}$ by the process of *differentiation.* For example:

$$\begin{aligned} y(x) &= 3x^2 + 4x - 1 \\ \Rightarrow \quad \frac{dy}{dx} &= 6x + 4 \end{aligned}$$

Integration is the reverse process:

When you know the derivative of a function, $\frac{dy}{dx}$, the process of finding the original function, y, is called *integration.*

For example:

$$\begin{aligned} \frac{dy}{dx} &= 10x - 3 \\ \Rightarrow \quad y(x) &= \ ? \end{aligned}$$

If you think about how differentiation works, you can probably see that the answer could be:

$$y = 5x^2 - 3x$$

However, there are many other possibilities; it could be $y = 5x^2 - 3x + 1$, or $y = 5x^2 - 3x - 20$, or ... in fact it could be any function of the form:

$$y = 5x^2 - 3x + c$$

where c is a constant. We say that $(5x^2 - 3x + c)$ is the *integral* of $(10x - 3)$ and write this as:

$$\underset{\substack{\text{The}\\ \text{integral}\\ \text{of}}}{\int} (10x - 3) \underset{\substack{\text{with}\\ \text{respect}\\ \text{to } x}}{dx} = 5x^2 - 3x + c$$

c is referred to as an "arbitrary constant" or a "constant of integration". More generally:

Integration is the reverse of differentiation.
If $f'(x)$ is the derivative of a function $f(x)$, then the integral of $f'(x)$ is $f(x)$ (plus an arbitrary constant):

$$\int f'(x)dx = f(x) + c$$

1.1. Integrating Powers and Polynomials

In the example above you can see that since differentiating powers of x involves reducing the power by 1, integrating powers of x must involve increasing the power by 1. The rule is:

Integrating Powers of x:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \qquad (n \neq -1)$$

It is easy to check that this rule works by differentiating:

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + c\right) = x^n$$

You can also see from this that the rule doesn't work when $n = -1$. But it works for other negative powers, for zero, and for non-integer powers – see the examples below.

We can apply this rule to integrate polynomials. For example:

$$\begin{aligned} \int (4x^2 + 6x - 3)dx &= \int (4x^2 + 6x - 3x^0)dx \\ &= 4 \times \frac{1}{3}x^3 + 6 \times \frac{1}{2}x^2 - 3x^1 + c \\ &= \frac{4}{3}x^3 + 3x^2 - 3x + c \end{aligned}$$

It is easy to make mistakes when integrating. You should always check your answer by differentiating it to make sure that you obtain the original function.

Examples 1.1: *Integrating Powers and Polynomials*

(i) What is the integral of $x^4 - 2x + 5$?

$$\int (x^4 - 2x + 5)dx = \frac{1}{5}x^5 - x^2 + 5x + c$$

(ii) Integrate $2 - \dfrac{t^5}{5}$.

$$\int \left(2 - \frac{t^5}{5}\right) dt = 2t - \frac{t^6}{30} + c$$

(iii) If $\dfrac{dy}{dx} = (2-x)(4-3x)$, what is y?

$$\begin{aligned} y &= \int (2-x)(4-3x)dx \\ &= \int (8-10x+3x^2)dx = 8x - 5x^2 + x^3 + c \end{aligned}$$

(iv) Integrate $1 + \dfrac{10}{z^3}$.

$$\begin{aligned} \int \left(1 + \frac{10}{z^3}\right) dz &= \int \left(1 + 10z^{-3}\right) dz \\ &= z + 10 \times \frac{1}{-2} z^{-2} + c \\ &= z - \frac{5}{z^2} + c \end{aligned}$$

(v) If $f'(x) = 3\sqrt{x}$ what is $f(x)$?

$$\begin{aligned} f(x) &= \int 3\sqrt{x}dx \\ &= \int 3x^{\frac{1}{2}}dx \\ &= 3 \times \frac{2}{3} x^{\frac{3}{2}} + c \\ &= 2x^{\frac{3}{2}} + c \end{aligned}$$

(vi) Integrate the function $3ax^2 + 2tx$ with respect to x.

$$\int (3ax^2 + 2tx)dx = ax^3 + tx^2 + c$$

(In this example there are several variables or parameters. We say "with respect to x" to clarify which one is to be treated as the variable of integration. The others are then treated as constants.)

Exercises 10.1: **Integrating Powers and Polynomials**

(1) Find: (a) $\int 8x^3 dx$ (b) $\int (2z - z^3 + 4)dz$ (c) $\int (1 + 3t^{-8})dt$

(d) $\int (a + bx)dx$ (e) $\int \left(\frac{q^2}{2} - \frac{18}{q^4}\right) dq$

(2) Integrate (a) $5x^{1.5}$ (b) $\sqrt{4z}$

(3) What is the integral of z^{3a-1} with respect to z?

(4) If $g'(p) = \alpha p^{\beta}$, what is $g(p)$?

1.2. Economic Application

Suppose we know that a firm's marginal cost of producing output y is $8y+3$, and also that the firm has a fixed cost of 10. Then we can integrate the marginal cost function to find the firm's total cost function:

$$\begin{aligned} C'(y) &= 8y+3 \\ \Rightarrow C(y) &= 4y^2+3y+c \end{aligned}$$

As usual, integration gives us an arbitrary constant, c. But in this case, we have another piece of information that tells us the value of c – the cost of producing zero output is 10:

$$\begin{aligned} C(0) &= 10 \quad \Rightarrow \quad c=10 \\ \Rightarrow C(y) &= 4y^2+3y+10 \end{aligned}$$

1.3. More Rules for Integration

Remember the rules for differentiating logarithmic and exponential functions (Chapter 6):

$$y = \ln x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$
$$y = e^{ax} \quad \Rightarrow \quad \frac{dy}{dx} = ae^{ax}$$

By reversing these we can obtain two more rules for integration:

$$\boxed{\begin{gathered} \int \frac{1}{x}\, dx = \ln x + c \\ \int e^{ax} dx = \frac{1}{a}\, e^{ax} + c \end{gathered}}$$

(The first of these rules tells us how to integrate x^n when $n{=}{-1}$, which we couldn't do before.)

Examples 1.2:

(i) $\displaystyle\int \left(6x + \frac{3}{x}\right) dx = \int \left(6x + 3 \times \frac{1}{x}\right) dx = 3x^2 + 3\ln x + c$

(ii) $\displaystyle\int \left(4e^{2x} + 15e^{-3x}\right) dx = 4 \times \frac{1}{2}e^{2x} + 15 \times \frac{1}{-3}e^{-3x} + c = 2e^{2x} - 5e^{-3x} + c$

Looking at the examples above you can also see that the following general rules hold. In fact they are obvious from what you know about differentiation, and you may have been using them without thinking about it.

$$\boxed{\begin{gathered} \int af(x)dx = a\int f(x)dx \\ \int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx \end{gathered}}$$

The rules for integrating $\frac{1}{x}$ and e^x can be generalised. From the Chain Rule for differentiation we can see that, if $f(x)$ is a function, then:

$$\frac{d}{dx}\left(\ln f(x)\right) = \frac{f'(x)}{f(x)} \quad \text{and} \quad \frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$

Reversing these gives us two further rules:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

To apply these rules you have to notice that an integral can be written in one of these forms, for some function $f(x)$.

Examples 1.3:

(i) $\int xe^{3x^2}dx$ We can rewrite this integral:

$$\begin{aligned}\int xe^{3x^2}dx &= \frac{1}{6}\int 6xe^{3x^2}dx \quad \text{and apply the } 2^{nd} \text{ rule above with } f(x) = 3x^2\\ &= \frac{1}{6}e^{3x^2} + c\end{aligned}$$

(ii) $\int \frac{1}{y+4}\,dx$ Applying the 1^{st} rule directly with $f(y) = y + 4$:

$$\int \frac{1}{y+4}\,dy = \ln(y+4) + c$$

(iii) $\int \frac{z+3}{z^2+6z-5}\,dz$

$$\begin{aligned}&= \tfrac{1}{2}\int \frac{2z+6}{z^2+6z-5}\,dz \qquad (f(z) = z^2 + 6z - 5)\\ &= \tfrac{1}{2}\ln(z^2+6z-5) + c\end{aligned}$$

Exercises 10.2: **Integrating Simple Functions**

(1) Find: (a) $\int 10e^{3x}dx$ (b) $\int \left(9y^2 - \frac{4}{y}\right)dy$ (c) $\int \left(\frac{1}{z} + \frac{1}{z^2}\right)dz$
(d) $\int t^2e^{-t^3}dt$ (e) $\int \frac{1}{2q-7}\,dq$

(2) If $f'(t) = 1 - e^{6t}$, what is $f(t)$?

(3) Integrate: $4x^2 - 2\sqrt{x} + 8x^{-3}$.

(4) If a firm has no fixed costs, and its marginal cost of producing output q is $(9q^{0.8} - 2)$, find the firm's total cost function $C(q)$.

(5) Find the integral with respect to x of $x^a + e^{ax} + x^{-a}$, assuming $-1 < a < 1$.

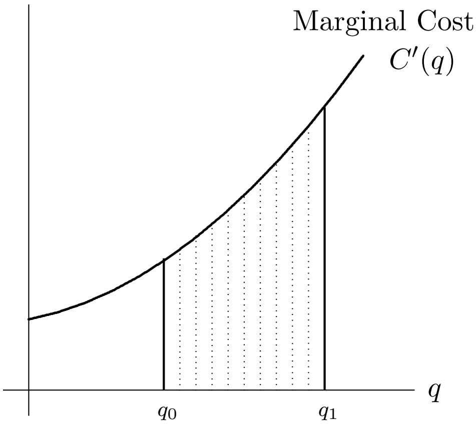
Further Reading and Exercises

- *Jacques* §6.1
- *Anthony & Biggs* §25.3

2. Integrals and Areas

In economics we often use areas on graphs to measure total costs and benefits: for example to evaluate the effects of imposing a tax. Areas on graphs can be calculated using integrals.

2.1. An Economic Example



The area under a firm's marginal cost curve between q_0 and q_1 represents the *total cost* of increasing output from q_0 to q_1: it adds up the marginal costs for each unit of output between q_0 and q_1.

So, the area represents: $C(q_1) - C(q_0)$.

To calculate this area we could:

- Integrate the marginal cost function $C'(q)$ to find the function $C(q)$
- Evaluate $C(q)$ at q_0 and q_1 to obtain $C(q_1) - C(q_0)$

Note that $C(q)$ will contain an arbitrary constant, but it will cancel out in $C(q_1) - C(q_0)$. (The constant represents the fixed costs – not needed to calculate the increase in costs.)

We write this calculation as:

$$\text{Area } = C(q_1) - C(q_0) = \int_{q_0}^{q_1} C'(q)dq$$

2.2. Definite Integration

$$\int_a^b f(x)dx$$

- represents the area under the graph of $f(x)$ between a and b.
- It is called a *definite integral.*
- a and b are called the *limits of integration.*
- To calculate it, we integrate, evaluate the answer at each of the limits, and subtract.

The type of integration that we did in the previous section is known as *indefinite* integration. For example, $\int (4x+1)dx = x^2 + x + c$ is an indefinite integral. The answer is a function of x

containing an arbitrary constant. In *definite* integration, in contrast, we evaluate the answer at the limits.

Examples 2.1: *Definite Integrals*

(i) $\int_{-1}^{4}(4x+1)dx$

This integral represents the area under the graph of the function $f(x)=4x+1$ between $x=-1$ and $x=4$.

$$\begin{aligned}\int_{-1}^{4}(4x+1)dx &= \left[2x^2+x+c\right]_{-1}^{4} \\ &= \left(2\times 4^2+4+c\right)-\left(2\times(-1)^2-1+c\right) \\ &= (36+c)-(1+c) \\ &= 35\end{aligned}$$

It is conventional to use square brackets here

From now on we will not bother to include the arbitrary constant in a definite integral since it always cancels out.

(ii) $\int_{9}^{25} 3\sqrt{y}\,dy$

$$\begin{aligned}\int_{9}^{25} 3\sqrt{y}\,dy &= \int_{9}^{25} 3y^{\frac{1}{2}}\,dy \\ &= \left[2y^{\frac{3}{2}}\right]_{9}^{25} \\ &= \left(2\times 25^{\frac{3}{2}}\right)-\left(2\times 9^{\frac{3}{2}}\right) \\ &= 250-54 \\ &= 196\end{aligned}$$

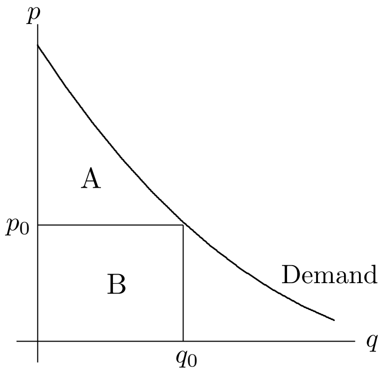
(iii) $\int_{1}^{a}\left(3+\frac{2}{q}\right)dq$ where a is a parameter.

$$\begin{aligned}\int_{1}^{a}\left(3+\frac{2}{q}\right)dq &= [3q+2\ln q]_{1}^{a} \\ &= (3a+2\ln a)-(3\times 1+2\ln 1) \\ &= (3a+2\ln a)-3 \\ &= 3a-3+2\ln a\end{aligned}$$

(iv) For a firm with marginal cost function $MC=3q^2+10$, find the increase in costs if output is increased from 2 to 6 units.

$$\begin{aligned}C(6)-C(2) &= \int_{2}^{6}(3q^2+10)dq \\ &= \left[q^3+10q\right]_{2}^{6} \\ &= \left(6^3+60\right)-\left(2^3+20\right) \\ &= 276-28 \\ &= 248\end{aligned}$$

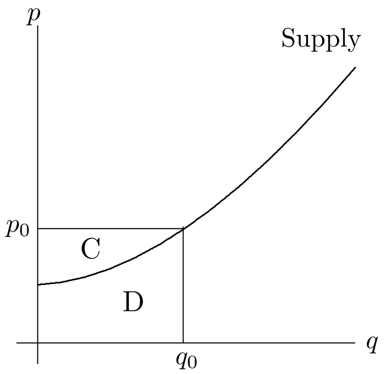
2.3. Economic Application: Consumer and Producer Surplus



The diagram shows a market demand function. When the market price is p_0, the quantity sold is q_0 and Area A represents net consumer surplus.

If the inverse demand function is $p = D(q)$, we can calculate Consumer Surplus by:

$$CS = \underbrace{\int_0^{q_0} D(q)dq}_{\text{Area(A+B)}} - \underbrace{p_0 q_0}_{\text{Area B}}$$



Similarly, Area C represents net producer surplus.

If the inverse supply function is $p = S(q)$, we can calculate Producer Surplus by:

$$PS = \underbrace{p_0 q_0}_{\text{Area(C+D)}} - \underbrace{\int_0^{q_0} S(q)dq}_{\text{Area D}}$$

EXERCISES 10.3: **Definite Integrals**

(1) Evaluate the definite integrals: (a) $\displaystyle\int_1^4 (2x^2+1)dx$ (b) $\displaystyle\int_{-1}^1 e^{3x}dx$

(2) For a market in which the inverse demand and supply functions are given by: $p^d(q) = 24 - q^2$ and $p^s(q) = q + 4$, find:
(a) the market price and quantity (b) consumer surplus (c) producer surplus.

(3) Evaluate the definite integrals:
$$\int_0^1 (x^2-3x+2)dx, \quad \int_1^2 (x^2-3x+2)dx, \quad \int_0^2 (x^2-3x+2)dx.$$
Explain the answers you obtain by sketching the graph of $y = x^2 - 3x + 2$.

Further Reading and Exercises

- *Jacques* §6.2. In particular §6.2.3 looks at investment, another economic application of definite integrals.
- *Anthony & Biggs* §§25.1, 25.2 and 25.4

3. Techniques for Integrating More Complicated Functions

The rules in sections 1.1 and 1.3 only allow us to integrate quite simple functions. They don't tell you, for example, how to integrate xe^x, or $\sqrt{3x^2+1}$.

Whereas it is possible to use rules to differentiate any "sensible" function, the same is not true for integration. Sometimes you just have to guess what the answer might be, then check whether you have guessed right by differentiating. There are some functions that cannot be integrated algebraically: the only possibility to is to use a computer to work out a numerical approximation.[1]

But when you are faced with a function that cannot be integrated by the simple rules, there are a number of techniques that you can try – one of them might work!

3.1. Integration by Substitution

Consider the integral:

$$\int (3x+1)^5 dx$$

We know how to integrate powers of x, but not powers of $(3x+1)$. So we can try the following procedure:

- Define a new variable: $t = 3x + 1$
- Differentiate: $\dfrac{dt}{dx} = 3 \quad \Rightarrow dt = 3dx \Rightarrow dx = \dfrac{1}{3}dt$
- Use these expressions to substitute for x and dx in the integral:

$$\int (3x+1)^5 dx = \int t^5 \frac{1}{3} dt$$

- Integrate with respect to t, then substitute back to obtain the answer as a function of x:

$$\begin{aligned} \int t^5 \frac{1}{3} dt &= \frac{1}{18} t^6 + c \\ &= \frac{1}{18}(3x+1)^6 + c \end{aligned}$$

You can check, by differentiating, that this is the integral of the original function. If you do this, you will see that integration by substitution is a way of reversing the Chain Rule (see Chapter 6).

Thinking about the Chain Rule, we can see that Integration by Substitution works for integrals that can be written in a particular form:

[1]The function $e^{-x^2/2}$, which is important in statistics, is an example.

When an integral can be written in the form:
$$\int f(t)\frac{dt}{dx}dx$$
where t is some function of x
then it can be integrated by substituting t for x:
$$\int f(t)\frac{dt}{dx}dx = \int f(t)dt$$

Sometimes you will be able to see immediately that an integral has the right form for using a substitution. Sometimes it is not obvious - but you can try a substitution and see if it works.

Note that the logarithmic and exponential rules that we found at the end of section 1.3 are a special case of the method of Integration by Substitution. The first example below could be done using those rules instead.

EXAMPLES 3.1: *Integration by substitution*

(i) $\displaystyle\int xe^{3x^2}dx$ For this integral we can use the substitution $t = 3x^2$:

$$t = 3x^2 \quad \Rightarrow dt = 6xdx$$

Substituting for x and dx:
$$\begin{aligned}\int xe^{3x^2}dx &= \int e^{3x^2}xdx \\ &= \int e^t\tfrac{1}{6}dt \\ &= \tfrac{1}{6}e^t + c = \tfrac{1}{6}e^{3x^2} + c\end{aligned}$$

(ii) $\displaystyle\int (3x^2+1)^5dx$

This example looks similar to the original one we did, but unfortunately the method does not work. Suppose we try substituting $t = 3x^2 + 1$:

$$t = 3x^2 + 1 \quad \Rightarrow dt = 6xdx$$

To substitute for x and dx we also need to note that $x = \sqrt{\dfrac{t-1}{3}}$. Then:

$$\int (3x^2+1)^5dx = \int t^5\frac{1}{6\sqrt{\frac{t-1}{3}}}dt$$

Doing the substitution has made the integral more difficult, rather than less.

(iii) $\displaystyle\int \frac{4y}{\sqrt{y^2-3}}dy$ Here we can use a substitution $t = y^2 - 3$:

$$t = y^2 - 3 \quad \Rightarrow dt = 2ydy$$

Substituting for y and dy:
$$\begin{aligned}\int \frac{4y}{\sqrt{y^2-3}}dy &= \int 2\frac{1}{\sqrt{y^2-3}}2ydy \\ &= \int 2t^{-\frac{1}{2}}dt \\ &= 4t^{\frac{1}{2}} + c = 4\sqrt{y^2-3} + c\end{aligned}$$

EXERCISES 10.4: **Integration by Substitution**

(1) Evaluate the following integrals using the suggested substitution:

(a) $\int x^2(x^3-5)^4 dx \qquad t = x^3 - 5$

(b) $\int (2z+1)e^{z(z+1)} dz \qquad t = z(z+1)$

(c) $\int \frac{3}{\sqrt{2y+3}} dy \qquad t = 2y+3$

(d) $\int \frac{x}{x^2+a} dx \qquad t = x^2 + a$ where a is a parameter

(e) $\int \frac{1}{(2q-1)^2} dq \qquad t = 2q-1$

(f) $\int \frac{p+1}{3p^2+6p-1} dp \qquad t = 3p^2 + 6p - 1$

(2) Evaluate the following integrals by substitution or otherwise:

(a) $\int (4x-7)^6 dx$ (b) $\int \frac{2q^3+1}{q^4+2q} dq$ (c) $\int xe^{-kx^2} dx$ (k is a parameter.)

3.2. Integration by Parts

Remember the Product Rule for differentiation:

$$\frac{d}{dx}\Big(u(x)v(x)\Big) = u(x)v'(x) + u'(x)v(x)$$

Integrating this we get:

$$u(x)v(x) = \int u(x)v'(x)dx + \int u'(x)v(x)dx$$

and rearranging gives us the following formula:

Integration by Parts:

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

Hence if we start with an integral that can be written in the form $\int u'(x)v(x)dx$ we can evaluate it using this rule provided that we know how to evaluate $\int u(x)v'(x)dx$.

EXAMPLES 3.2: *Integration by Parts*

(i) $\int xe^{5x} dx$ Suppose we let $v(x) = x$ and $u'(x) = e^{5x}$.

$$\begin{array}{rclcrcl} u'(x) & = & e^{5x} & \Rightarrow & u(x) & = & \frac{1}{5}e^{5x} \\ v(x) & = & x & \Rightarrow & v'(x) & = & 1 \end{array}$$

Applying the formula:

$$\begin{aligned}\int u'(x)v(x)dx &= u(x)v(x) - \int u(x)v'(x)dx\\ \int xe^{5x}dx &= \tfrac{1}{5}xe^{5x} - \int \tfrac{1}{5}e^{5x}dx\\ &= \tfrac{1}{5}xe^{5x} - \tfrac{1}{25}e^{5x} + c\end{aligned}$$

(Check this result by differentiation.)

(ii) $\int (4y+1)(y+2)^3 dy$ Let $v(y) = 4y+1$ and $u'(y) = (y+2)^3$:

$$\begin{aligned}u'(y) &= (y+2)^3 &\Rightarrow\quad u(y) &= \tfrac{1}{4}(y+2)^4\\ v(y) &= 4y+1 &\Rightarrow\quad v'(y) &= 4\end{aligned}$$

(If this is not obvious to you it could be done by substitution)

Applying the formula:

$$\begin{aligned}\int u'(y)v(y)dy &= u(y)v(y) - \int u(y)v'(y)dy\\ \int (4y+1)(y+2)^3dy &= \tfrac{1}{4}(4y+1)(y+2)^4 - \int (y+2)^4dy\\ &= \tfrac{1}{4}(4y+1)(y+2)^4 - \tfrac{1}{5}(y+2)^5 + c\end{aligned}$$

(iii) $\int \ln x\, dx$

This is a standard function, but cannot be integrated by any of the rules we have so far. Since we don't have a product of two functions, integration by parts does not seem to be a promising technique. However, if we put:

$$\begin{aligned}u'(x) &= 1 &\Rightarrow\quad u(x) &= x\\ v(x) &= \ln x &\Rightarrow\quad v'(x) &= \tfrac{1}{x}\end{aligned}$$

and apply the formula, we obtain:

$$\begin{aligned}\int u'(x)v(x)dx &= u(x)v(x) - \int u(x)v'(x)dx\\ \int \ln x\, dx &= x\ln x - \int x\frac{1}{x}dx\\ &= x\ln x - \int 1.dx\\ &= x\ln x - x + c\end{aligned}$$

Again, check by differentiating.

Exercises 10.5: Use integration by parts to find:

(1) $\int (x+1)e^x dx$ (2) $\int 2y\ln y\, dy$

3.3. Integration by Substitution and by Parts: Definite Integrals

3.3.1. *Integration by Substitution.* If we want to find:

$$\int_0^1 \frac{2x}{x^2+8}dx$$

we could integrate by substituting $t = x^2+8$ to find the indefinite integral, and then evaluate it at the limits. However, a quicker method is to substitute for the limits as well:

$$\begin{aligned} t = x^2+8 &\Rightarrow dt = 2xdx \\ x = 0 &\Rightarrow t = 8 \\ x = 1 &\Rightarrow t = 9 \end{aligned}$$

Hence:

$$\begin{aligned} \int_0^1 \frac{2x}{x^2+8}dx &= \int_0^1 \frac{1}{x^2+8}2xdx \\ &= \int_8^9 \frac{1}{t}dt \\ &= [\ln t]_8^9 \\ &= \ln 9 - \ln 8 = \ln 1.125 \end{aligned}$$

3.3.2. *Integration by Parts.* Similarly, the method of integration by parts can be modified slightly to deal with definite integrals. The formula becomes:

$$\int_a^b u'(x)v(x)dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u(x)v'(x)dx$$

Consider, for example: $\displaystyle\int_0^1 (1-x)e^x\,dx$

We can integrate by parts using: $\begin{array}{rclcrcl} u'(x) &=& e^x &\Rightarrow& u(x) &=& e^x \\ v(x) &=& 1-x &\Rightarrow& v'(x) &=& -1 \end{array}$

Applying the formula:

$$\begin{aligned} \int_0^1 (1-x)e^x\,dx &= \Big[(1-x)e^x\Big]_0^1 + \int_0^1 e^x dx \\ &= -1 + \big[e^x\big]_0^1 \\ &= e-2 \end{aligned}$$

Exercises 10.6: **Definite Integrals**

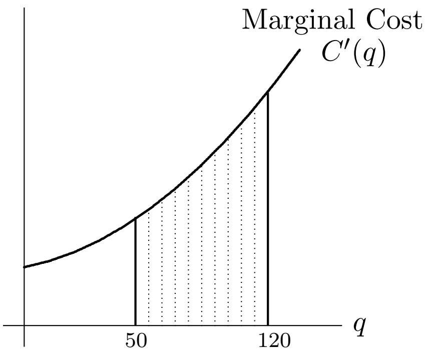
(1) Integrate by substitution: (a) $\displaystyle\int_0^2 (2x+1)^3dx$ (b) $\displaystyle\int_{-1}^1 e^{1+3y}dy$

(2) Integrate by parts: $\displaystyle\int_0^1 4ze^{-2z}\,dz$

Further Reading and Exercises

- *Anthony & Biggs* §§26.1, 26.2 and 26.3

4. Integrals and Sums



Suppose a firm with marginal cost curve $C'(q)$ increases its output from 50 units to 120. The total increase in cost is given by the integral of the marginal cost curve:

$$C(120) - C(50) = \int_{50}^{120} C'(q)dq$$

We can think of this as the sum of all the marginal costs of the units of output between 50 and 120, so:

$$\int_{50}^{120} C'(q)dq \approx C'(51) + C'(52) + \cdots + C'(120)$$

So we can see that:

$$\int_{50}^{120} C'(q)dq \approx \sum_{q=51}^{100} C'(q)$$

These two expressions are only approximately equal because we have a *continuous* marginal cost curve, allowing for fractions of units of output.

In general, we can think of integrals as representing the equivalent of a *sum*, used when we are dealing with continuous functions.

4.1. Economic Application: The Present Value of an Income Flow

Remember from Chapter 3 that when interest is compounded continously at annual rate i, the present value of an amount A received in t years time is given by:

$$Ae^{-it}$$

Suppose you receive an annual income of y for T years, and that, just as interest is compounded continuously, your income is paid continuously. This means that in a short time period length Δt, you will receive $y\Delta t$. For example, in a day ($\Delta t = \frac{1}{365}$) you would get $\frac{1}{365}y$. In an infinitesimally small time period dt your income will be:

$$ydt$$

which has present value:

$$ye^{-it}dt$$

Then the present value of the whole of your income stream is given by the "sum" over the whole period:

$$V = \int_0^T ye^{-it}dt$$

We can calculate the present value in this way even if the income stream is not constant – that is, if $y = y(t)$.

Examples 4.1: *The Present Value of an Income Flow*

(i) An investment will yield a constant continuous income of £1000 per year for 8 years. What is its present value if the interest rate is 2%?

$$\begin{aligned} V &= \int_0^8 1000e^{-0.02t}dt \\ &= 1000\left[-\frac{1}{0.02}e^{-0.02t}\right]_0^8 \\ &= 50000(-e^{-0.16}+1) = £7393 \end{aligned}$$

(ii) A worker entering the labour market expects his annual earnings, y, to grow continuously according to the formula $y(t) = £12000e^{0.03t}$ where t is length of time that he has been working, measured in years. He expects to work for 40 years. If the interest rate is $i = 0.05$, what is the present value of his expected lifetime earnings?

$$\begin{aligned} V &= \int_0^{40} y(t)e^{-it}dt \\ &= \int_0^{40} 12000e^{0.03t}e^{-0.05t}dt \\ &= 12000\int_0^{40} e^{-0.02t}dt \\ &= 12000\left[-\frac{1}{0.02}e^{-0.02t}\right]_0^{40} \\ &= 600000\left[-e^{-0.02t}\right]_0^{40} \\ &= 600000\left(-e^{-0.8}+1\right) = £330403 \end{aligned}$$

Exercises 10.7: **The present value of an income flow**

(1) What is the present value of a constant stream of income of £200 per year for 5 years, paid continuously, if the interest rate is 5%?

(2) A worker earns a constant continuous wage of w per period. Find the present value, V, of his earnings if he works for T periods and the interest rate is r. What is the limiting value of V as $T \to \infty$?

Further Reading and Exercises

- *Jacques* §6.2.4.
- *Anthony & Biggs* §§25.1, 25.2 and 25.4

Solutions to Exercises in Chapter 10

Exercises 10.1:

(1) (a) $2x^4 + c$
 (b) $z^2 - \frac{1}{4}z^4 + 4z + c$
 (c) $t - \frac{3}{7}t^{-7} + c$
 (d) $ax + \frac{1}{2}bx^2 + c$
 (e) $\frac{q^3}{6} + \frac{6}{q^3} + c$

(2) (a) $2x^{2.5} + c$
 (b) $\frac{4}{3}z^{3/2} + c$

(3) $\frac{1}{3a}z^{3a} + c$

(4) $\frac{\alpha}{\beta+1}p^{\beta+1} + c$

Exercises 10.2:

(1) (a) $\frac{10}{3}e^{3x} + c$
 (b) $3y^3 - 4\ln y + c$
 (c) $\ln z - \frac{1}{z} + c$
 (d) $-\frac{1}{3}e^{-t^3} + c$
 (e) $\frac{1}{2}\ln(2q-7) + c$

(2) $f(t) = t - \frac{1}{6}e^{6t} + c$

(3) $\frac{4}{3}x^3 - \frac{4}{3}x^{3/2} - 4x^{-2} + c$

(4) $C(q) = 5q^{1.8} - 2q$

(5) $\frac{1}{a+1}x^{a+1} + \frac{1}{a}e^{ax} + \frac{1}{1-a}x^{1-a} + c$

Exercises 10.3:

(1) (a) $\left[\frac{2}{3}x^3 + x\right]_1^4 = 45$
 (b) $\left[\frac{1}{3}e^{3x}\right]_{-1}^1 = \frac{1}{3}(e^3 - e^{-3}) = 6.68$

(2) (a) $q = 4$, $p = 8$
 (b) $\left[24q - \frac{1}{3}q^3\right]_0^4 - 32 = 42\frac{2}{3}$
 (c) $32 - \left[\frac{1}{2}q^2 + 4q\right]_0^4 = 8$

(3) $\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^1 = \frac{5}{6}$
 $\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_1^2 = -\frac{1}{6}$
 $\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^2 = \frac{2}{3}$
 Between $x = 1$ and $x = 2$ the graph is below the x-axis. So the area between the graph and the axis here is negative.

Exercises 10.4:

(1) (a) $dt = 3x^2dx$
 $\int t^4\frac{1}{3}dt = \frac{1}{15}t^5 + c = \frac{1}{15}(x^3-5)^5 + c$
 (b) $dt = (2z+1)dz$
 $\int e^t dt = e^t + c = e^{z(z+1)} + c$
 (c) $dt = 2dy$
 $\int \frac{3}{2t^{\frac{1}{2}}}dt = 3t^{\frac{1}{2}} + c = 3\sqrt{2y+3} + c$
 (d) $dt = 2xdx$
 $\int \frac{1}{2t}dt = \frac{1}{2}\ln t + c = \frac{1}{2}\ln(x^2+a) + c$
 (e) $dt = 2dq$
 $\int \frac{1}{2t^2}dt = -\frac{1}{2t} + c = -\frac{1}{4q-2} + c$
 (f) $dt = (6p+6)dp = 6(p+1)dp$
 $\int \frac{1}{t}\frac{1}{6}dt = \frac{1}{6}\ln t + c$
 $= \frac{1}{6}\ln(3p^2 + 6p - 1) + c$

(2) (a) $t = 4x - 7$, $dt = 4dx$
 $\frac{1}{28}(4x-7)^7 + c$
 (b) $t = q^4 + 2q$, $dt = (4q^3+2)dq$
 $\frac{1}{2}\ln(q^4 + 2q) + c$
 (c) $t = \pm kx^2$, $dt = \pm 2kxdx$
 $-\frac{1}{2k}e^{-kx^2} + c$

Exercises 10.5:

(1) $u' = e^x$, $v = x + 1$
 $xe^x + c$

(2) $u' = 2y$, $v = \ln y$
 $y^2\ln y - \frac{1}{2}y^2 + c$

Exercises 10.6:

(1) (a) $t = 2x + 1 \Rightarrow \frac{1}{8}\left[t^4\right]_1^5 = 78$
 (b) $t = 1 + 3y \Rightarrow \frac{1}{3}\left[e^t\right]_{-2}^4 = \frac{1}{3}\left(e^4 - e^{-2}\right)$

(2) $u' = e^{-2z}$, $v = 4z \Rightarrow$
 $\left[-2ze^{-2z}\right]_0^1 + \left[-e^{-2z}\right]_0^1 = 1 - 3/e^2$

Exercises 10.7:

(1) $\int_0^5 200e^{-0.05t}dt = 200\left[-\frac{1}{0.5}e^{-0.05t}\right]_0^5$
 $= 4000(1 - e^{-0.25}) = £884.80$

(2) $V = \int_0^T we^{-rt}dt$
 $= w\left[-\frac{1}{r}e^{-rt}\right]_0^T = \frac{w}{r}\left(1 - e^{-rT}\right)$
 $V \to \frac{w}{r}$ as $T \Rightarrow \infty$

[2]This Version of Workbook Chapter 10: September 15, 2003

Worksheet 10: Integration, and Further Optimisation Problems

Integration

(1) Evaluate the following integrals:

$$\text{(a)} \int (x-3)(x+1)dx \qquad \text{(b)} \int_0^1 e^{-5y}dy \qquad \text{(c)} \int \frac{2z+3}{z^2}dz$$

(2) Evaluate the following integrals, using a substitution if necessary:

$$\text{(a)} \int (3x+1)^9 dx \qquad \text{(b)} \int \frac{y-3}{y^2-6y+1}dy \qquad \text{(c)} \int_0^1 \frac{z^3}{\sqrt{z^2+1}}dz$$

(3) A competitive firm has inverse supply function $p = q^2 + 1$ and fixed costs $F = 20$. Find its total cost function.

(4) An investment will yield a continuous profit flow $\pi(t)$ per year for T years. Profit at time t is given by:

$$\pi(t) = a + bt$$

where a and b are constants. If the interest rate is r, find the present value of the investment. (Hint: you can use integration by parts.)

(5) The inverse demand and supply functions in a competitive market are given by:

$$p^d(q) = \frac{72}{1+q} \text{ and } p^s(q) = 2 + q$$

(a) Find the equilibrium price and quantity, and consumer surplus.
(b) The government imposes a tax $t = 5$ on each unit sold. Calculate the new equilibrium quantity, tax revenue, and the deadweight loss of the tax.

Further Problems

(1) An incumbent monopoly firm Alpha faces the following market demand curve:

$$Q = 96 - P,$$

where Q is the quantity sold per day, and P is the market price. Alpha can produce output at a constant marginal cost of £6, and has no fixed costs.

(a) What is the price Alpha is charging? How much profit is it making per day?
(b) Another firm, Beta, is tempted to enter the market given the high profits that the incumbent, Alpha, is making. Beta knows that Alpha has a cost advantage: if it enters, its marginal costs will be twice as high as Alpha's, though there will be no fixed costs of entry. If Beta does enter the market, is expects Alpha to act as a Stackelberg leader (i.e. Beta maximizes its profits taking Alpha's output as given; Alpha maximizes its profits taking into account that Beta will react in this way). Show that under these assumptions, Beta will find it profitable to enter, despite the cost disadvantage. How much profit would each firm earn?
(c) For a linear demand curve of the form $Q = a - bP$, show that consumers' surplus is given by the expression $CS = \frac{1}{2b}Q^2$. Evaluate the benefit to consumers of increased competition in the market once Beta has entered.

(2) A monopolist knows that to sell x units of output she must charge a price of $P(x)$, where $P'(x) < 0$ and $P''(x) < 0$ for all x. The monopolist's cost of producing x is $C(x) = A + ax^2$, where A and a are both positive numbers. Let x^* be the firm's profit-maximizing output. Write down the first- and second-order conditions that x^* must satisfy. Show that x^* is a decreasing function of a. How does x^* vary with A?

(3) An Oxford economic forecasting firm has the following cost function for producing reports:

$$C(y) = 4y^2 + 16$$

where y is the number of reports.

(a) What are its average and marginal cost functions?
(b) At what number of reports is its average cost minimized?
(c) Initially the market for economic forecasts in Oxford is extremely competitive and the going price for a report is £15. Should the firm continue to produce reports? Why or why not?
(d) Suppose the price rises to £20. How many reports will the firm supply? Illustrate diagrammatically and comment on your answer.
(e) Suddenly all competitor forecasting firms go out of business. The demand for reports is such that $p = 36 - 6y$, where p is the price of a report. How many reports will the Oxford firm produce? What profit will it earn?

(4) The telecommunications industry on planet Mercury has an inverse demand curve given by $P = 100 - Q$. The marginal cost of a unit of output is 40 and fixed costs are 900. Competition in this industry is as in the Cournot model. There is free entry into and exit from the market.

(a) How many firms survive in equilibrium?
(b) Due to technical advance, fixed costs fall to 400 from 900. What happens to the number of firms in the industry?
(c) What is the effect on price of the fall in fixed costs? Explain your answer.

(5) Students at St. Gordon's College spend all their time in the college bar drinking and talking on their mobile phones. Their utility functions are all the same and are given by $U = DM - O^2$, where D is the amount of drink they consume, M is the amount of time they talk on their mobile phones and O is the amount of time each other student spends on the phone, over which they have no control. Both drink and mobile phone usage cost £1 per unit, and each student's income is £100.

(a) How much time does each student spend on the phone and how much does each drink?
(b) What is the utility level of each student?
(c) The fellows at the college suggest that mobile phone usage should be taxed at 50 pence per unit, the proceeds from the tax being used to subsidize the cost of fellows' wine at high table. Are the students better off if they accept this proposal?
(d) The economics students at the college suggest that the optimal tax rate is twice as large. Are they right? What other changes might the students propose to improve their welfare?