C2 Jan 06

1. Use the Trapezium Rule with six ordinates to find an approximate value for the integral $\int_0^1 \frac{1}{2+x^3}dx$

Show your working and give your answer correct to four significant figures. [4]

Solution

$$h = \frac{1-0}{5} = 0.2$$

x			
0	$\frac{1}{2+0^3} = 0.5$		$= 0.5$
0.2	$\frac{1}{2+0.2^3} = 0.498007668$	$\times 2$	$= 0.996015336$
0.4	$\frac{1}{2+0.4^3} = 0.484496124$	$\times 2$	$= 0.968992248$
0.6	$\frac{1}{2+0.6^3} = 0.451263537$	$\times 2$	$= 0.902527074$
0.8	$\frac{1}{2+0.8^3} = 0.398089172$	$\times 2$	$= 0.796178344$
1	$\frac{1}{2+1^3} = 0.3333333333$		$= 0.33333333$
		sum	$= 4.497046335$

$$Area \approx \frac{0.2}{2}[4.497046335] = 0.4497 \ \ (4 \ \ d.p.)$$

2. Find all values of θ in the interval $0° \le \theta \le 360°$ satisfying

(a) $4\cos^2\theta - \cos\theta = 2\sin^2\theta$ [6]

(b) $\tan\theta = -\sqrt{3}$ [2]

(c) $\sin 2\theta = \frac{1}{2}$ [3]

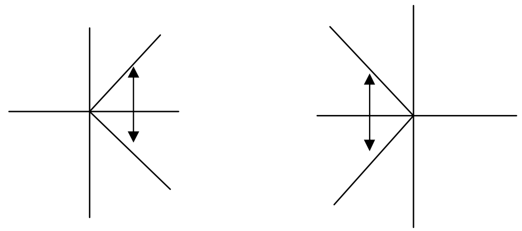
Solution

(a) $4\cos^2\theta - \cos\theta = 2\sin^2\theta$ and using $\sin^2\theta = 1-\cos^2\theta$ we get

$$4\cos^2\theta - \cos\theta = 2(1-\cos^2\theta)$$
$$4\cos^2\theta - \cos\theta = 2-2\cos^2\theta$$
$$6\cos^2\theta - \cos\theta - 2 = 0$$
Factorising gives
$$(3\cos\theta-2)(2\cos\theta+1) = 0$$
$$either \ 3\cos\theta - 2 = 0, \ \ 2\cos\theta + 1 = 0$$
$$\cos\theta = \frac{2}{3}, \ \ \cos\theta = -\frac{1}{2}$$

Acute angles ignoring -ve signs gives

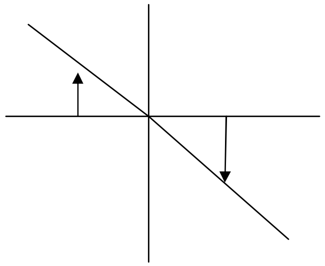
$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2, \ \ \ \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°$$



$$\theta = 48.2, \ 311.8, \ 120, \ 240°$$

(b) Acute angle ignoring -ve sign is

$\theta = \tan^{-1}\sqrt{3} = 60°$



$\theta = 120,\ 300°$

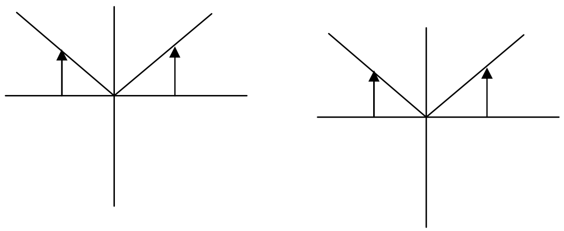
(c) $\sin 2\theta = -\frac{1}{2}$

Adjusted Range

$0 \le \theta \le 360$
$(\times 2) \quad 0 \le 2\theta \le 720°$

Acute angle ignoring the sign

$2\theta = \sin^{-1}\left(\frac{1}{2}\right) = 30°$



$2\theta = 30,\ 150,\ \ 390,\ \ 510$
$\theta = 15,\ 75,\ 195,\ 255°$

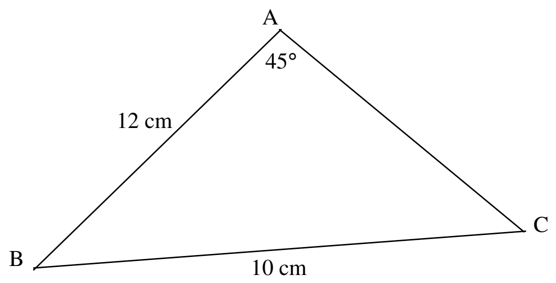
3. The triangle ABC is such that AB is 12 cm, BC is 10 cm and $C\hat{A}B = 45°$.

(a) Find the possible values of $B\hat{C}A$ and $A\hat{B}C$. [4]

(b) Find the possible values of the area of the triangle ABC. [2]

<u>Solution</u>

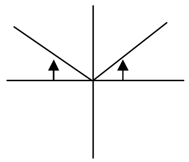
(a)



Two sides and two angles are involved therefore we use the Sine Rule

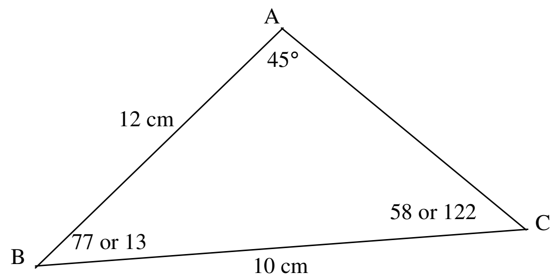
$$\frac{a}{SinA} = \frac{b}{SinB}$$
$$\frac{10}{\sin 45} = \frac{12}{\sin B\hat{C}A}$$
$$\sin B\hat{C}A = \frac{12 \times \sin 45}{10} = 0.84853$$
$$B\hat{C}A = \sin^{-1} 0.84853 = 58°$$

or for other angle possible



$x = 180 - 58 = 122°$

(b)



$A\hat{C}B = 180 - (45 + 58) = 77°$ or

$A\hat{C}B = 180 - (45 + 122) = 13°$

$Area = \frac{1}{2} \times a \times b \times \sin C = \frac{1}{2} \times 12 \times 10 \times \sin 77° = 58.5\ cm^2$ (1 $d.p.$) or

$Area = \frac{1}{2} \times a \times b \times \sin C = \frac{1}{2} \times 12 \times 10 \times \sin 13° = 13.5\ cm^2$ (1 $d.p.$)

4. (a) A geometric sereis has first term a and common ratio r. Write down the nth term of the series and prove that the sum of the first n terms is given by $S_n = \frac{a(1-r^n)}{1-r}$ [4]

(b) The fourth term of a geometric series is 2 and the seventh term is 54.

(i) Find the common ratio of the series.

(ii) Find the sum of the first ten terms of the series, giving your answer correct to one decimal place.

(iii) Find the least value of the nth term to exceed 125000. [10]

Solution

(a) See proof in notes

(b) (i) $T_4 = ar^3 = 2....[1]$ $\qquad T_7 = ar^6 = 54....[2]$

$[2] \div [1]$ gives

$$\frac{ar^6}{ar^3} = \frac{54}{2} = 27$$
$$r^3 = 27$$
$$r = 3$$

(ii) As $ar^3 = 2$

$$a(3)^3 = 2$$
$$a = \frac{2}{27}$$

$$S_{10} = \frac{\frac{2}{27}\left(1-3^{10}\right)}{1-3} = 2186.96.... = 2187.0 \ (1\ d.p.)$$

(iii) We need $ar^{n-1} > 125000$ with $a = \frac{2}{27}$ and $r = 3$

$$\frac{2}{27}(3)^{n-1} > 125000$$
$$(\times 27 \div 2) \quad 3^{n-1} > 1687500$$

We have an unknown power and therefore use logs!!!!

$$\log 3^{n-1} > \log 1687500$$
$$(n-1)\log 3 > \log 1687500$$
$$n-1 > \frac{\log 1687500}{\log 3}$$
$$n > \frac{\log 1687500}{\log 3} + 1$$
$$n > 14.05.....$$
$$n = 15$$

5. The sum of the first two terms of an arithemetic series is 3. The eighth terms of the arithmetic series is 47.

Find

(a) the first term and the common difference of the series. [4]

(b) the sum of the first twenty terms of the series. [2]

Solution

(a) $T_1 + T_2 = 3$

$$a + a + d = 3$$
$$2a + d = 3.....[1]$$

$$T_8 = a + 7d = 47.....[2]$$

Solving simultaneously we get

[1] - 2[2]

$$\begin{aligned} 2a + d &= 3 \\ -\underline{2a + 14d} &\underline{= 94} \\ -13d &= -91 \\ d &= 7 \end{aligned}$$

From [1] $2a + d = 3$

$$2a + 7 = 3$$
$$2a = -4$$
$$a = -2$$

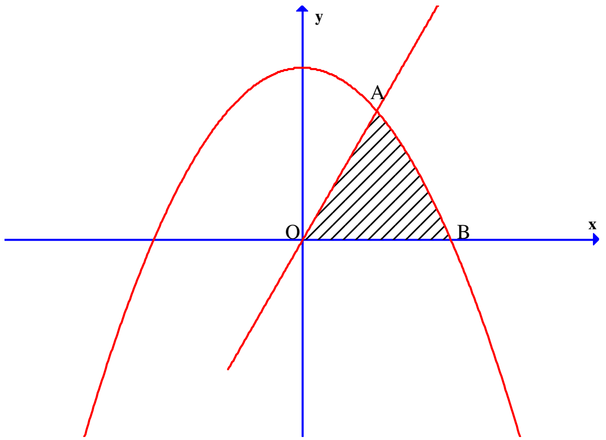
$$S_{20} = \frac{20}{2}[2(-2) + 19(7)] = 1290$$

6. Integrate $5x^{\frac{1}{3}} + 3x^{-3}$ with respect to x. [2]

Solution

$$\int 5x^{\frac{1}{3}} + 3x^{-3}dx = \frac{5x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{3x^{-2}}{-2} + c = \frac{3}{4} \times 5x^{\frac{4}{3}} - \frac{3}{2x^2} + c$$
$$= \frac{15}{4}x^{\frac{4}{3}} - \frac{3}{2x^2} + c$$

7.



The diagram shows the curve $y = 4 - x^2$ and the line $y = 3x$ intersecting at the point A. The curve $y = 4 - x^2$ intersects the x-axis at B.

(a) Find the coordinates of A and B, showing your working. [5]

(b) Evaluate the area of the shaded region. [7]

Solution

(a) A and B can be found by

A- solving the two equations simultaneously

B-letting $y = 0$ and solving $y = 4 - x^2$ and taking the positive x value.

For A Point of intersection

$$4 - x^2 = 3x$$
$$0 = x^2 + 3x - 4$$
$$0 = (x+4)(x-1)$$
$$x = -4,\ 1$$

So A is $(1, 3)$ as $y = 3x$

For B

$$0 = 4 - x^2$$
$$x^2 = 4$$
$$x = \pm 2$$

So B is (2, 0)

(b) Area under line is $\int_0^1 3x dx = \left[\frac{3x^2}{2}\right]_0^1 = \left(\frac{3}{2}\right) - (0) = \frac{3}{2}$

Area under curve is $\int_1^2 4 - x^2 dx = \left[4x - \frac{x^3}{3}\right]_1^2 = \left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)$

$= \frac{5}{3}$

Shaded area is $\frac{3}{2} + \frac{5}{3} = 3\frac{1}{6}$ $units^2$

8. (a) Find the radius and centre of the circle C given by

$$x^2 + y^2 - 8x + 4y + 11 = 0$$

(b) Given that the circle [3]

$$x^2 + y^2 = a^2 \quad (a > 0)$$

touches C externally, find the value of a, giving your answer correct to two decimal places. [4]

Solution

(a) Complete the square on the x's and y's

$$x^2 - 8x \ + y^2 + 4y + 11 = 0$$
$$(x-4)^2 + (y+2)^2 - 4 + 11 = 0$$
$$(x-4)^2 + (y+2)^2 = 9$$
$$centre = (4, -2) \quad radius = 3$$

(b) As they touch externally distance between centres equals sum of radii!

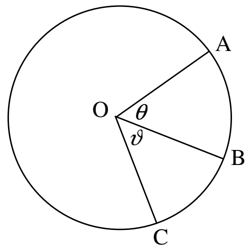
For $x^2 + y^2 = a^2$ the centre is (0, 0) and the radius is a.

Distance between centres is $\sqrt{(4-0)^2 + (-2-0)^2} = \sqrt{20}$

So $\sqrt{20} = 3 + a$ and squaring both sides gives

$$a = \sqrt{20} - 3 = 1.47 \ \ (2 \ \ d.p.)$$

9.



The diagram shows a circle of centre O and radius 4 cm. The points A, B and C lie on the circle as shown. The angles θ and φ being measured in radians. The sum of the sector areas AOB and BOC is 15.2 cm^2.

(a) Show that $\theta + \vartheta = 1.9$ [2]

(b) Given that the arc length AB is 3.2 cm greater than the arc length BC, find the values of θ and ϑ [4]

Solution

(a) Sum of sectors = $\frac{1}{2}r^2\phi + \frac{1}{2}r^2\theta = 15.2 \ cm^2$

$r = 4$ and so $\frac{1}{2}4^2\phi + \frac{1}{2}4^2\theta = 15.2 \ cm^2$

$$8\phi + 8\theta = 15.2$$
$$(\div 8) \quad \phi + \theta = 1.9......[1]$$

(b) Arc length AB = 3.2 +Arc length BC

$$r\theta = 3.2 + r\phi$$
$$r = 4 \quad 4\theta = 3.2 + 4\phi$$
$$4\theta - 4\phi = 3.2$$
$$(\div 4) \quad \theta - \phi = 0.8......[2]$$

Now solving simultaneously [1] + [2]

$$\begin{aligned} 2\theta &= 2.7 \\ \theta &= 1.35 \end{aligned} \text{ and so}$$

$\theta - \phi = 0.8$
$1.35 - \phi = 0.8$
$\phi = 0.55$

10. (a) Given that $x > 0, \quad y > 0$ show that

$$\log_a (xy) = \log_a x + \log_a y$$ [3]

(b) Given that $\int_1^3 \log_{10} x dx$ has an approximate value of 0.5628,

find an approximate value for $\int_1^3 \log_{10} xdx$. Given your answer correct to four decimal places. [4]

Solution

(a) See proof in notes

(b) $\int_1^3 \log_{10} 10x = \int_1^3 \log_{10} 10dx + \int_1^3 \log_{10} xdx$ using above rule!!!!

$= \int_1^3 1dx + 0.5628$
$= [x]_1^3 + 0.5628$
$= (3) - (1) + 0.5628$
$= 2.5628 \ (4 \ d.p.)$