C2 May 09

1. Use the Trapezium Rule with five ordinates to find an approximate value for the integral $\int_0^{0.4} \frac{1}{2+\sqrt{x}}\,dx$.

Show your working and give your answer correct to three decimal places. [4]

<u>Solution</u>

$$h = \frac{0.4-0}{4} = 0.1$$

x	$\frac{1}{2+\sqrt{x}}$		
0	$\frac{1}{2+\sqrt{0}} = 0.5$		$= 0.5$
0.1	$\frac{1}{2+\sqrt{0.1}} = 0.4317364703$	$\times 2$	$= 0.8634729406$
0.2	$\frac{1}{2+\sqrt{0.2}} = 0.4086280012$	$\times 2$	$= 0.8172560024$
0.3	$\frac{1}{2+\sqrt{0.3}} = 0.3935074169$	$\times 2$	$= 0.7850148338$
0.4	$\frac{1}{2+\sqrt{0.4}} = 0.3798734633$		$= 0.3798734633$
		sum	$= 3.34561724$

$$Area \approx \frac{0.1}{2}[3.34561724] = 0.167 \quad (3\ d.p.)$$

2. (a) Find all values of θ between $0°$ and $360°$ satisfying

$$5\cos^2\theta + 2 = 3\sin^2\theta - 2\cos\theta$$ [6]

(b) Find all value of x between $0°$ and $180°$ satisfying

$$\sin(2x+12°) = -0.53$$ [3]

<u>Solution</u>

(a) Replace the squared trig function that does not have a non-squared trig function present! i.e. $\sin^2\theta$ by $1-\cos^2\theta$

$$5\cos^2\theta + 2 = 3(1-\cos^2\theta) - 2\cos\theta$$

$5\cos^2\theta + 2 = 3 - 3\cos^2\theta - 2\cos\theta$ and moving all terms to the RHS gives

$8\cos^2\theta + 2\cos\theta - 1 = 0$ and factorising gives

$(4\cos\theta - 1)(2\cos\theta + 1) = 0$ and so

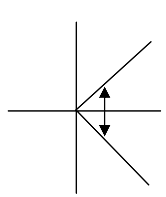
either $4\cos\theta - 1 = 0$ or $2\cos\theta + 1 = 0$

and so $\cos\theta = \frac{1}{4}$, $\cos\theta = -\frac{1}{2}$

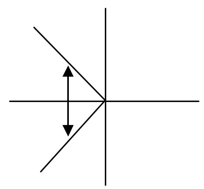
Acute angle ignoring sign are

$\theta = \cos^{-1}\frac{1}{4} = 75.5°$ $\qquad$ $\theta = \cos^{-1}\frac{1}{2} = 60°$

And taking account of the signs give



$\theta = 75.5, 284.5°$



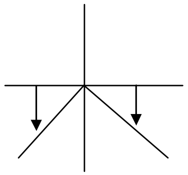
$\theta = 104.5, 255.5°$

(b) Adjusted range

$0° \le x \le 180°$
$(\times 2) \quad 0° \le 2x \le 360°$
$(+12) \quad 12° \le 2x+12 \le 372°$

Acute angle ignoring the sign is

$2x+12 = \sin^{-1} 0.53 = 32.0°$



$2x+12° = 212°, \qquad 2x+12° = 328°$
$2x = 200, \qquad 2x = 316$
$x = 100°, \quad x = 158°$

(both within the range) Check for others!!

3. The triangle ABC is such that AB = 16 cm, AC = 9 cm and $A\hat{B}C = 23°$.

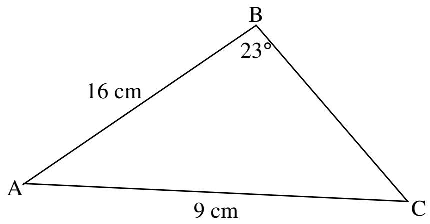
(a) Find the possible values of $A\hat{C}B$. Give you answers correct to the nearest degree. [2]

(b) Given that $B\hat{A}C$ is an **acute** angle, find

(i) the size of $B\hat{A}C$, giving your answer correct to the nearest degree,

(ii) the area of the triangle ABC, giving your answer correct to one decimal place. [4]

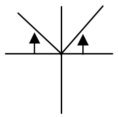
<u>Solution</u>



(a) Using the Sine Rule we get

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{9}{\sin 23} = \frac{16}{\sin A\hat{C}B}$$
$$\sin A\hat{C}B = \frac{\sin 23 \times 16}{9}$$
$$= 0.6946$$

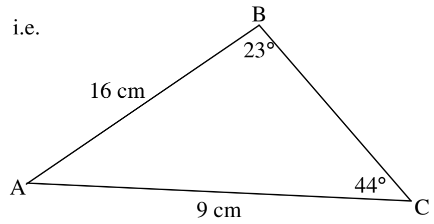
Acute angle $A\hat{C}B = \sin^{-1} 0.6946 = 44$ *to nearest degree*



$A\hat{C}B = 44, \ 136°$

(b) If this angle is acute it is $44°$

i.e.



(i) Using the fact that angles in a triangle add up to 180 degrees we can clearly see that
$B\hat{A}C = 180 - 23 - 44 = 113°$

(ii) Area = $\frac{1}{2}ab\sin C = \frac{1}{2}\times 16\times 9\times \sin 113 = 66.3\ cm^2$

4. (a) An arithmetic series has first term a and common difference d. Prove at the sum of the first n terms of the series is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$ [3]

(b) The eighth term of an arithmetic series is 46. The sum of the first nine terms of the series is 225. Find the first term and the common difference the the series. [4]

(c) Find an expression, in terms of n, for the sum of the first n terms of the arithmetic series 3 + 7 +11 + 11 + 15 + ...

Simplify your answer. [3]

Solution

(a) See notes for proof

(b) $T_8 = a + 7d = 46.....[1]$

$$S_9 = \frac{9}{2}[2a + 8d] = 225$$
$$4.5[2a + 8d] = 225$$
$$(\div 4.5)\ \ 2a + 8d = 50$$
$$(\div 2)\ \ a + 4d = 25.......[2]$$

Solving [1] and [2] simultaneously

$$\begin{aligned} a + 7d &= 46 \\ -\ a + 4d &= 25 \\ \hline 3d &= 21 \\ d &= 7 \end{aligned}$$

Now as $d = 7$ and using [1]

$$a + 7(7) = 46$$
$$a = -3$$

(b) $a = 3,\ d = T_n - T_{n-1} = 7 - 3 = 4$

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ &= \frac{n}{2}[2(3) + (n-1)4] \\ &= \frac{n}{2}[6 + 4n - 4] \\ &= \frac{n}{2}[4n + 2] \\ &= n[2n + 1] \end{aligned}$$

5. (a) The ninth and tenth terms of a geometric series are 36 and 108 respectively. Find the seventh term of the series.

(b) Another geometric series has first term a and common ratio r. The second term of this geometric series is 9 and the sum to infinity of the series is 48.

(i) Show that r satisfies the equation $16r^2 - 16r + 3 = 0$

(ii) Find the two possible values for r and the corresponding values for a.

<u>Solution</u>

(a) $T_9 = ar^8 = 36.....[1]$
$T_{10} = ar^9 = 108....[2]$

We could solve simultaneously by dividing but as they are consecutive terms we can just divide the second by the first to get r!!!!

i.e. $r = \frac{108}{36} = 3$

To get term 7 we divide term 8 by the common ratio! i.e. 3

$T_7 = \frac{36}{3} = 12$

(b) $T_2 = ar = 9 ....a = \frac{9}{r}$....[1]

$S_\infty = \frac{a}{1-r} = 48 ....a = 48(1-r)$.....[2]

Solving [1] and [2] simultaneously

$\frac{9}{r} = 48(1-r)$
$9 = 48r(1-r)$
$9 = 48r - 48r^2$
$(\div 3)\ \ 3 = 16r - 16r^2$
and moving to the LHS gives

$16r^2 - 16r + 3 = 0$

Factorising we get

$(4r-3)(4r-1) = 0$
either $4r - 3 = 0$ *or* $4r - 1 = 0$
$r = \frac{3}{4}$ *or* $\frac{1}{4}$

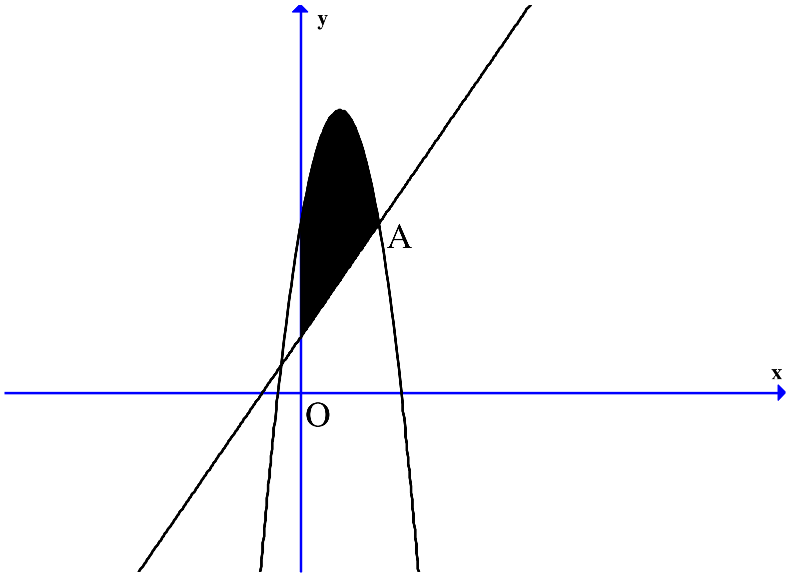
Using $a = \frac{9}{r}$....[1]

If $r = \frac{3}{4}$, $a = \frac{9}{\frac{3}{4}} = 9 \times \frac{4}{3} = 12$

$r = \frac{1}{4}$, $a = \frac{9}{\frac{1}{4}} = 4 \times 9 = 36$

6. (a) Find $\int \left(\frac{5}{x^3} - 3x^{\frac{1}{4}} \right) dx$ [2]

(b)



The diagram shows a sketch of the curve $y = 6 + 4x - x^2$ and the line $y = x + 2$. The point of intersection of the curve and the line in the first quadrant is denoted by A.

(i) Find the coordinates of A.

(ii) Find the area of the shaded region. [10]

<u>Solution</u>

(a) $\int \left(\frac{5}{x^3} - 3x^{\frac{1}{4}} \right) dx = \int \left(5x^{-3} - 3x^{\frac{1}{4}} \right) dx = \frac{5x^{-2}}{-2} - \frac{3x^{\frac{5}{4}}}{\frac{5}{4}}$

$= -\frac{5}{2x^2} - \frac{4}{5} \times 3x^{\frac{5}{4}} = -\frac{5}{2x^2} - \frac{12}{5} x^{\frac{5}{4}} + c$

(b) (i) The coordinates of A is one of the points on intersection. Need to solve the two equations simultaneously

$$6+4x-x^2 = x+2$$
$$0 = x^2-3x-4$$
$$0 = (x-4)(x+1)$$
$$x = -1,\ 4$$

Therefore as $y = x+2$

A is (4, 6)

(ii) The area under the curve is

$$\int_0^4 (6+4x-x^2)dx = \left[6x+2x^2-\frac{x^3}{3}\right]_0^4 \left(24+32-\frac{64}{3}\right)-(0) = \frac{104}{3} = 34\frac{2}{3}$$

The area under the line is $\int_0^4 (x+2)dx = \left[\frac{x^2}{2}+2x\right]_0^4 = (8+8)-(0) = 16$

Therefore the shaded area = curve - line = $34\frac{2}{3}-16 = 18\frac{2}{3}\ units^2$

7. (a) Given that $x > 0,\ y > 0$, show that

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$ [3]

(b) Solve the equation

$$3^{5-2x} = 7$$

Show your working and give your answer correct to three decimal places. [3]

(c) Solve the equation

$$\log_a (x-3) + \log_a (x+3) = 2\log_a (x-2)$$ [4]

Solution

(a) See proof in notes

(b) $3^{5-2x} = 7$ and taking logs

$$\log_a 3^{5-2x} = \log_a 7$$
$$(5-2x)\log_a 3 = \log_a 7$$
$$5-2x = \frac{\log_a 7}{\log_a 3}$$
$$5-2x = 1.771244$$
$$-2x = -3.22876$$
$$x = 1.614\ (3\ d.p.)$$

(c)
$$\log_a (x-3) + \log_a (x+3) = 2\log_a (x-2)$$
$$\log_a (x-3) + \log_a (x+3) = \log_a (x-2)^2$$
$$\log_a (x-3)(x+3) = \log_a (x-2)^2$$

By comparing we get

$$(x-3)(x+3) = (x-2)^2$$
$$x^2-9 = x^2-4x+4$$
$$4x = 13$$
$$x = \frac{13}{4}$$

8. The circle C_1 has centre A and equation

$$x^2 + y^2 - 6x + 2y - 15 = 0$$

(a) Find the coordinates of A and the radius of C_1 [3]

(b) The point P has coordinates (7, 2) and lies on C_1. Find the equation of the tangent to C_1 at P. [4]

(c) The circle C_2 has centre B(11, 14) and radius 8. A point Q lies on C_1 and a point R lies on C_2. Find the shortest possible length of the line QR. [3]

<u>Solution</u>

(a) $x^2 + y^2 - 6x + 2y - 15 = 0$ and rearranging gives

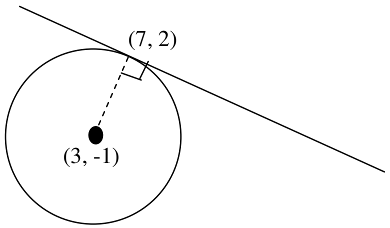
$x^2 - 6x + y^2 + 2y - 15 = 0$ and completing the square gives

$$(x-3)^2 - 9 + (y+1)^2 - 1 - 15 = 0$$

$$(x-3)^2 + (y+1)^2 = 25$$

Therefore centre is (3, -1) and the radius is 5

(b)



First of all we need the gradient of the tangent. We work out the gradient of the radius (shown) and the gradient of the tangent is -1 over this!

$$gradient\ of\ radius = \frac{7-3}{2--1} = \frac{4}{3}$$

Therefore the gradient of the tangent is $\dfrac{-1}{\frac{4}{3}} = -\dfrac{3}{4}$

and so the equation of the tangent is

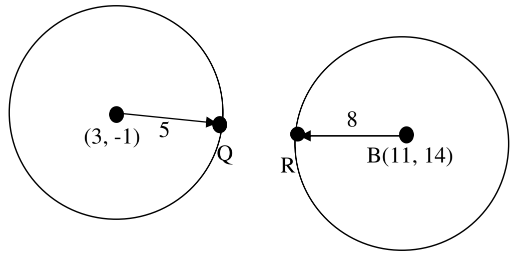
$$y - 2 = -\frac{3}{4}(x-7)$$

$$(\times 4)\ \ 4(y-2) = -3(x-7)$$

$$4y - 8 = -3x + 21$$

$$3x + 4y = 29$$

(c)

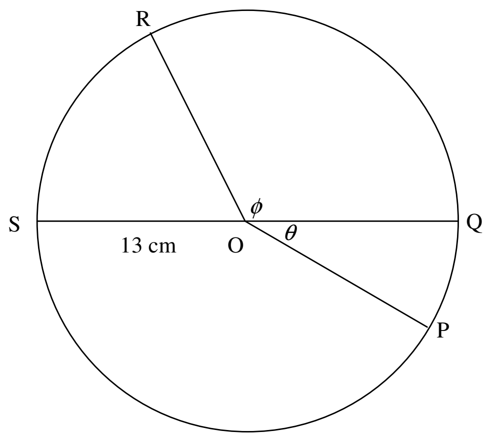


The shortest length of the line QR is given by the distance between the centres minus each of the radii!!!.

Distance between centres = $\sqrt{(11-3)^2 + (14--1)^2} = 17$

Therefore the shortest distance is $17 - 5 - 8 = 4$

9.



The diagram shows the four points P, Q, R and S on a circle with centre O and radius 13 cm. The line QS is a diameter of the circle, $P\hat{O}Q = \theta$ *radians* and $Q\hat{O}R = \phi$ radians.

(a) The area of the sector POQ is $60cm^2$. Find the value of θ, giving your answer correct to two decimal places. [2]

(b) The length of the arc QR is 7 cm greater than the length of the arc PS. Find the value of ϕ, giving your answer correct to two decimal places. [3]

<u>Solution</u>

(a) Area of Sector = $\frac{1}{2}r^2\theta = 60$ and as $r = 13$

$$\frac{1}{2}\times 13^2\theta = 60$$

$$\theta = \frac{60}{84.5} = 0.71^c$$

(b) Arc QR = $r\phi = 13\phi$

Arc RS = $r(\pi - \phi) = 13(\pi - \phi)$

Given that Length of Arc QR = 7 + Length of Arc RS

$$13\phi = 7 + 13(\pi - \phi)$$
$$13\phi = 7 + 13\pi - 13\phi$$
$$26\phi = 7 + 13\pi$$
$$\phi = \frac{7 + 13\pi}{26}$$
$$\phi = 1.84^c$$