Question 9

$$\vec{r}=(1-\lambda-\mu)\vec{a}+\lambda\vec{b}+\mu\vec{c}=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right)+\mu\left(\vec{c}-\vec{a}\right)=\overrightarrow{OA}+\lambda\overrightarrow{AB}+\mu\overrightarrow{AC}$$

This is clearly a plane through the points A, B and C .

Now $\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)=\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

Since $\vec{r}-\vec{a}$ exists in the plane $\left(\vec{r}-\vec{a}\right)\bullet\vec{n}=0$

$$\Rightarrow\left(\vec{r}-\vec{a}\right)\bullet\left(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right)=0$$

$$\Rightarrow\left((-\lambda-\mu)\vec{a}+\lambda\vec{b}+\mu\vec{c}\right)\bullet\left(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right)=0$$

Further $\begin{cases}\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\end{cases}\Rightarrow\vec{a}\bullet\vec{b}\times\vec{c}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

This means that in any triple product of this form, if the same vector occurs twice then the triple product is equal to zero.

Therefore $(-\lambda-\mu)\vec{a}\bullet\vec{b}\times\vec{c}+\lambda\vec{b}\bullet\vec{c}\times\vec{a}+\mu\vec{c}\bullet\vec{a}\times\vec{b}=0$ because all the other triple products involve the duplication of one of the components.

Since this relation holds for all λ,μ choosing :-

$$\lambda=1,\mu=-1\Rightarrow\vec{b}\bullet\vec{c}\times\vec{a}=\vec{c}\bullet\vec{a}\times\vec{b}$$

$$\lambda=1,\mu=0\Rightarrow\vec{b}\bullet\vec{c}\times\vec{a}=\vec{a}\bullet\vec{b}\times\vec{c}$$

Therefore $\vec{a}\bullet\vec{b}\times\vec{c}=\vec{b}\bullet\vec{c}\times\vec{a}=\vec{c}\bullet\vec{a}\times\vec{b}$

If the points O, A, B, C all lie in the same plane then trivially:-

$$\vec{a}\bullet\vec{b}\times\vec{c}=\vec{b}\bullet\vec{c}\times\vec{a}=\vec{c}\bullet\vec{a}\times\vec{b}=0$$