QUESTION 5

Let $x = f(t), y = g(t)$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{g(t)}{f(t)}\right) \Rightarrow \frac{d\theta}{dt} = \frac{1}{1+\left(\frac{g(t)}{f(t)}\right)^2} \times \frac{f(t)g'(t)-g(t)f'(t)}{f^2(t)}$$

$$= \frac{1}{f^2(t)+g^2(t)} \times \left(f(t)g'(t)-g(t)f'(t)\right)$$

$$= \frac{1}{r^2} \times \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

Hence $\frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$

The coordinates of A are $(x - a\cos t, y - a\sin t)$

The coordinates of B are $(x + b\cos t, y + b\sin t)$

$$[A] = \frac{1}{2}\int_0^{2\pi}\left\{(x - a\cos t)\left(\frac{dy}{dt} - a\cos t\right) - (y - a\sin t)\left(\frac{dx}{dt} + a\sin t\right)\right\}dt$$

$$[P] = \frac{1}{2}\int_0^{2\pi}\left\{x\frac{dy}{dt} - y\frac{dx}{dt}\right\}dt$$

Hence $[A] = \frac{1}{2}\int_0^{2\pi}\left\{\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) - a\left(\left(x + \frac{dy}{dt}\right)\cos t + \left(y - \frac{dx}{dt}\right)\sin t\right) + a^2\right\}dt$

$$= [P] + \frac{1}{2}\int_0^{2\pi} a^2 dt - a\left\{\frac{1}{2}\int_0^{2\pi}\left(\left(x + \frac{dy}{dt}\right)\cos t + \left(y - \frac{dx}{dt}\right)\sin t\right)dt\right\}$$

$$= [P] + \pi a^2 - af \quad (f \text{ as defined})$$

$$[B] = \frac{1}{2}\int_0^{2\pi}\left\{(x + b\cos t)\left(\frac{dy}{dt} + b\cos t\right) - (y + b\sin t)\left(\frac{dx}{dt} - b\sin t\right)\right\}dt$$

Hence $\left[B\right]=\frac{1}{2}\int_0^{2\pi}\left\{\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)+b\left(\left(x+\frac{dy}{dt}\right)\cos t+\left(y-\frac{dx}{dt}\right)\sin t\right)+b^2\right\}dt$

$$=\left[P\right]+\frac{1}{2}\int_0^{2\pi}b^2dt+b\left\{\frac{1}{2}\int_0^{2\pi}\left(\left(x+\frac{dy}{dt}\right)\cos t+\left(y-\frac{dx}{dt}\right)\sin t\right)dt\right\}$$

$$=\left[P\right]+\pi b^2+bf \ \ (f\ \text{as defined})$$

Therefore $\pi a^2-af=\left[A\right]-\left[P\right]=\left[B\right]-\left[P\right]=\pi b^2+bf$

$$\Rightarrow \pi\left(a^2-b^2\right)=\left(a+b\right)f$$

$$\Rightarrow f=\pi\left(a-b\right)$$

Hence $\left[A\right]-\left[P\right]=\pi a^2-a\pi\left(a-b\right)=\pi ab$
