

QUESTION 7

$$\text{Let } \theta = \frac{2\pi}{23} \text{ and } \begin{cases} C = \cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos 21\theta \\ S = \sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin 21\theta \end{cases}$$

$$\begin{aligned} \text{Then } C + iS &= \sum_{r=1}^{11} e^{(2r-1)i\theta} = \frac{e^{i\theta}(1 - e^{22i\theta})}{1 - e^{2i\theta}} = \frac{1 - e^{22i\theta}}{e^{-i\theta} - e^{i\theta}} = \frac{e^{11i\theta}(e^{-11i\theta} - e^{11i\theta})}{e^{-i\theta} - e^{i\theta}} \\ &= \frac{e^{11i\theta} \times (-2i \sin 11\theta)}{-2i \sin \theta} = \frac{e^{11i\theta} \sin 11\theta}{\sin \theta} \\ &= \frac{\sin 11\theta}{\sin \theta} (\cos 11\theta + i \sin 11\theta) \end{aligned}$$

$$\text{Therefore } S = \frac{\sin^2 11\theta}{\sin \theta} = \frac{\sin^2 \frac{22\pi}{23}}{\sin \frac{2\pi}{23}} = \frac{\sin \frac{22\pi}{23} \times \sin \frac{22\pi}{23}}{2 \sin \frac{\pi}{23} \cos \frac{\pi}{23}} = \frac{\sin \frac{\pi}{23}}{2 \cos \frac{\pi}{23}} = \frac{1}{2} \tan \frac{\pi}{23}$$

$$\text{Again let } \theta = \frac{2\pi}{23} \text{ and } \begin{cases} C = \cos \theta - \cos 3\theta + \cos 5\theta - \dots + \cos 21\theta \\ S = \sin \theta - \sin 3\theta + \sin 5\theta - \dots + \sin 21\theta \end{cases}$$

$$\begin{aligned} \text{Then } C + iS &= \sum_{r=1}^{11} (-1)^{r+1} e^{(2r-1)i\theta} = \frac{e^{i\theta}(1 - e^{22i\theta})}{1 + e^{2i\theta}} = \frac{1 - e^{22i\theta}}{e^{-i\theta} + e^{i\theta}} = \frac{e^{11i\theta}(e^{-11i\theta} - e^{11i\theta})}{e^{-i\theta} + e^{i\theta}} \\ &= \frac{e^{11i\theta} \times (-2i \sin 11\theta)}{2 \cos \theta} = \frac{-ie^{11i\theta} \sin 11\theta}{\cos \theta} \\ &= \frac{\sin 11\theta}{\cos \theta} (\sin 11\theta - i \cos 11\theta) \end{aligned}$$

$$\begin{aligned}
 \text{Therefore } S &= -\frac{\sin 11\theta \cos 11\theta}{\cos \theta} = \frac{-\sin \frac{22\pi}{23} \cos \frac{22\pi}{23}}{\cos \frac{2\pi}{23}} = -\frac{\sin \frac{44\pi}{23}}{2 \cos \frac{2\pi}{23}} \\
 &= \frac{\sin \frac{2\pi}{23}}{2 \cos \frac{2\pi}{23}} = \frac{1}{2} \tan \frac{2\pi}{23}
 \end{aligned}$$
