S2 May 2012 Solutions.

1. A manufacturer produces sweets of length L mm where L has a continuous uniform distribution with range [15, 30].

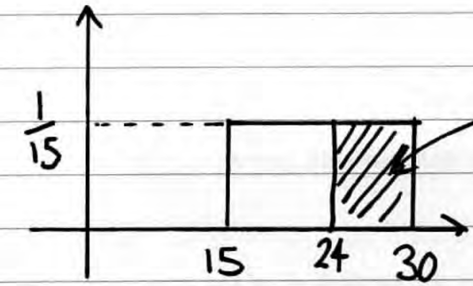
(a) Find the probability that a randomly selected sweet has a length greater than 24 mm. **(2)**

These sweets are randomly packed in bags of 20 sweets.

(b) Find the probability that a randomly selected bag will contain at least 8 sweets with length greater than 24 mm. **(3)**

(c) Find the probability that 2 randomly selected bags will both contain at least 8 sweets with length greater than 24 mm. **(2)**

a)



$P(L>24) = 6 \times \frac{1}{15} = \underline{\underline{0.4}}$

b) X = no. of sweets with $L>24$

$X \sim B(20, 0.4)$

$P(X \geq 8) = 1 - P(X \leq 7)$

$= 1 - 0.4159 = \underline{\underline{0.5841}}$

c) $(0.5841)^2 = \underline{\underline{0.3412}}$ (4dp)

2. A test statistic has a distribution B(25, p).

Given that

$$H_0 : p = 0.5 \qquad H_1 : p \neq 0.5$$

(a) find the critical region for the test statistic such that the probability in each tail is as close as possible to 2.5%.

(3)

(b) State the probability of incorrectly rejecting H_0 using this critical region.

(2)

a) under H_0 $X \sim B(25, 0.5)$

$P(X \leq 7) = 0.0216$ $\qquad C_1 = 7$

$P(X \leq 8) = 0.0539$

$P(X \geqslant 18) = 1 - P(X \leq 17)$
$= 1 - 0.9784 = 0.0215$ $\qquad C_2 = 18$

$P(X \geqslant 17) = 1 - P(X \leq 16)$
$= 1 - 0.9461 = 0.0539$

Critical region: $X \leq 7,\ X \geqslant 18$

b) $0.0215 + 0.0216 = 0.0431$

3. (a) Write down two conditions needed to approximate the binomial distribution by the Poisson distribution.

(2)

A machine which manufactures bolts is known to produce 3% defective bolts. The machine breaks down and a new machine is installed. A random sample of 200 bolts is taken from those produced by the new machine and 12 bolts were defective.

(b) Using a suitable approximation, test at the 5% level of significance whether or not the proportion of defective bolts is higher with the new machine than with the old machine. State your hypotheses clearly.

(7)

a) n is large
p is small (mean $\approx$ variance)

b) X = no. of defective bolts

$H_0 : p = 0.03 \quad H_1 : p > 0.03$

under H_0 $X \sim B(200, 0.03)$

$X \approx \sim Po(6)$

$P(X \geq 12) = 1 - P(X \leq 11)$

$= 1 - 0.9799$

$= 0.0201$

$0.0201 < 0.05$ so reject H_0

There is evidence that the proportion of defective bolts is higher with the new machine.

4. The number of houses sold by an estate agent follows a Poisson distribution, with a mean of 2 per week.

(a) Find the probability that in the next 4 weeks the estate agent sells,

(i) exactly 3 houses,

(ii) more than 5 houses.

(5)

The estate agent monitors sales in periods of 4 weeks.

(b) Find the probability that in the next twelve of these 4 week periods there are exactly nine periods in which more than 5 houses are sold.

(3)

The estate agent will receive a bonus if he sells more than 25 houses in the next 10 weeks.

(c) Use a suitable approximation to estimate the probability that the estate agent receives a bonus.

(6)

X = no. of houses sold

a) $X \sim Po(8)$

$P(X=3) = \frac{e^{-8} 8^3}{3!} = \underline{\underline{0.0286}}$

$P(X>5) = 1 - P(X \leq 5)$

$= 1 - 0.1912 = \underline{\underline{0.8088}}$

b) Y = no. of periods where $X>5$

$Y \sim B(12, 0.8088)$

$P(Y=9) = \binom{12}{9} (0.8088)^9 (0.1912)^3$

$= \underline{\underline{0.2277}}$

Question 4 continued

c) $X \sim Po(20)$ approximate by $W \sim N(20, \sqrt{20}^2)$

$P(X > 25) = P(X \geqslant 26) \approx P(W \geqslant 25.5)$

$= P\left(Z \geqslant \frac{25.5 - 20}{\sqrt{20}}\right)$

$= P(Z \geqslant 1.23)$

$= 1 - \Phi(1.23)$

$= \underline{\underline{0.1093}}$

5. The queueing time, X minutes, of a customer at a till of a supermarket has probability density function

$$f(x) = \begin{cases} \frac{3}{32}x(k-x) & 0 \leqslant x \leqslant k \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that the value of k is 4
(4)

(b) Write down the value of E(X).
(1)

(c) Calculate Var(X).
(4)

(d) Find the probability that a randomly chosen customer's queueing time will differ from the mean by at least half a minute.
(3)

a) $\int_0^k \frac{3}{32}x(k-x)\,dx = 1$

$\frac{3}{32}\int_0^k (kx-x^2)\,dx = 1$

$\cancel{\frac{3}{32}}\left[\frac{kx^2}{2}-\frac{x^3}{3}\right]_0^k = \frac{32}{3}$

$\frac{k^3}{2}-\frac{k^3}{3} = \frac{32}{3}$

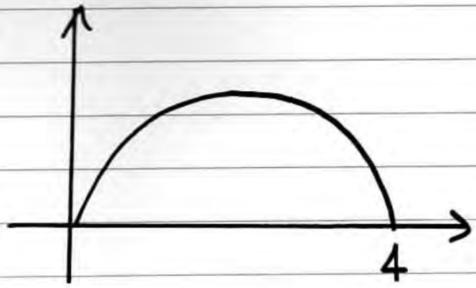
$\frac{k^3}{6} = \frac{32}{3}$

$k^3 = 64$

$\underline{k = 4}$

Question 5 continued

b)



$E(X) = \underline{\underline{2}}$

c) $E(X^2) = \int_0^4 \frac{3}{32} x^3 (4-x)\,dx$

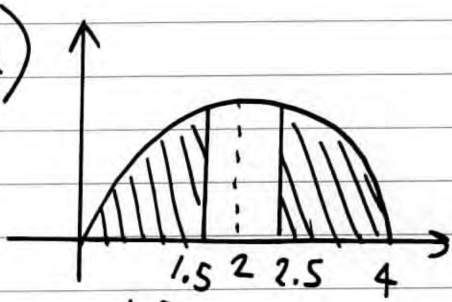
$= \frac{3}{32}\left[x^4 - \frac{x^5}{5}\right]_0^4$

$= \frac{3}{32}\left[256 - \frac{1024}{5}\right]$

$= \frac{24}{5}$

$Var(X) = \frac{24}{5} - 2^2 = \underline{\underline{0.8}}$

d)



$2\int_0^{1.5} \frac{3}{32} x(4-x)\,dx = \frac{3}{16}\left[2x^2 - x^3/3\right]_0^{1.5}$

$= \frac{3}{16} \times \frac{27}{8} = \frac{81}{128}$ //.