STEP Further Maths Paper B, July 1993, Q13

QUESTION

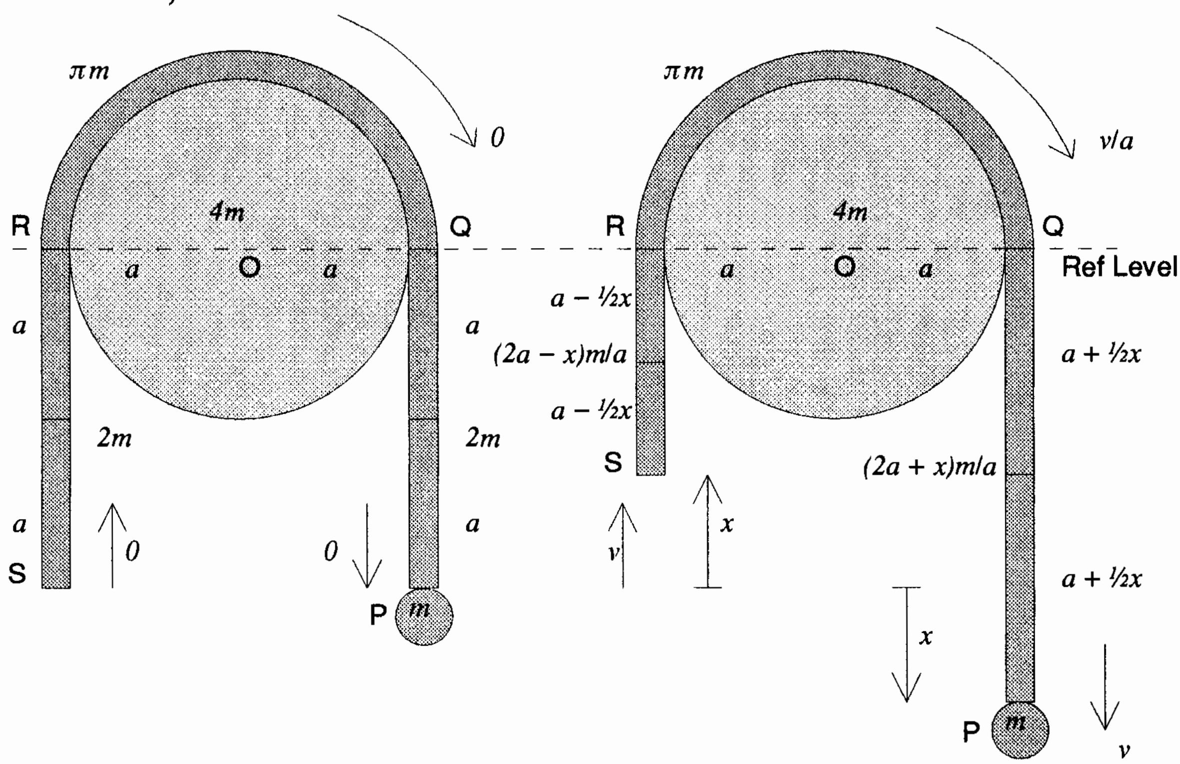
A uniform circular disc with radius a, mass $4m$ and centre O is freely mounted on a fixed horizontal axis which is perpendicular to its plane and passes through O. A uniform heavy chain PS of length $(4 + \pi)a$, mass $(4 + \pi)m$ and negligible thickness is hung over the rim of the disc as shown in the diagram; Q and R are the points of the chain at the same level as O. The contact between the chain and the rim of the disc is sufficiently rough to prevent slipping. Initially, the system is at rest with $PQ = RS = 2a$. A particle of mass m is attached to the chain at P and the system is released. By considering the energy of the system, show that when P has descended a distance x, its speed v is given by

$$(\pi + 7)av^2 = 2g(x^2 + ax).$$

By considering the part PQ of the chain as a body of variable mass, show that when S reaches R the tension in the chain at Q is

$$\frac{5\pi - 2}{\pi + 7}mg.$$

ANSWER, First Part



C of E:

$\frac{1}{2}mv^2 + \frac{1}{2}(\frac{1}{2}(4m)a^2(\frac{v}{a})^2) + \frac{1}{2}(4 + \pi)mv^2 - (a - \frac{1}{2}x)g(2a - x)\frac{m}{a} - (a + \frac{1}{2}x)g(2a + x)\frac{m}{a}$
$- mg(2a + x) = 2mg(-a) + 2mg(-a) + mg(-2a)$

$\therefore v^2 + 2v^2 + (4 + \pi)v^2 - (2a - x)^2\frac{g}{a} - (2a + x)^2\frac{g}{a} - 2g(2a + x) = -12ga$

$\therefore (\pi + 7)av^2 = (4a^2 - 4ax + x^2)g + (4a^2 + 4ax + x^2)g + 4ga^2 + 2gax - 12ga^2$

$\therefore (\pi + 7)av^2 = 2g(ax + x^2),$ *as desired.* (i)

ANSWER, Second Part, First Method

$\frac{d}{dx}(i)$: $\quad (\pi + 7)a2v\frac{dv}{dx} = 2g(a + 2x)$

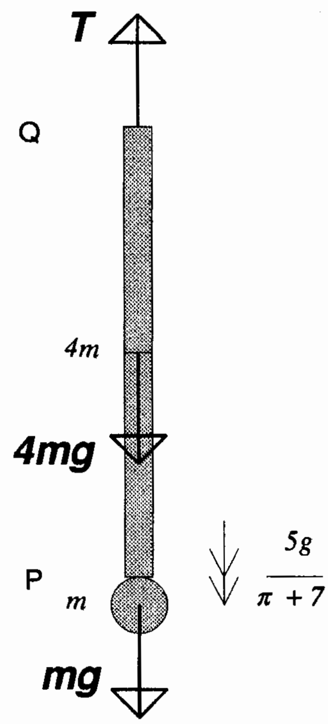
$\therefore \quad v\frac{dv}{dx} = \frac{(a+2x)g}{(\pi+7)a}$

When $x = 2a$, $\quad v\frac{dv}{dx} = \frac{5g}{\pi+7}$

Newton $\downarrow$: $5mg - T = 5m\frac{5g}{\pi+7}$

$\therefore \quad T = 5mg(1 - \frac{5}{\pi+7})$

$\therefore \quad T = 5mg(\frac{2+\pi}{\pi+7})$ (ANS)



ANSWER, Second Part, Second Method

Let energy of QP and the particle be E. Then

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\tfrac{m}{a}(2a + x)v^2 - (a + \tfrac{1}{2}x)g(2a + x)\tfrac{m}{a} - mg(2a + x)$$

$$= \tfrac{3}{2}mv^2 + \tfrac{1}{2}\tfrac{m}{a}xv^2 - \tfrac{mg}{2a}(2a + x)^2 - mg(2a + x)$$

C of E: $\frac{dE}{dt} = \frac{1}{2}(\frac{m}{a}v)v^2 - Tv$

$\therefore \quad 3mv\frac{dv}{dt} + \frac{1}{2}\frac{m}{a}vv^2 + \frac{m}{a}xv\frac{dv}{dt} - \frac{mg}{a}(2a + x)v - mgv = \frac{1}{2}\frac{m}{a}v^3 - Tv$

$\therefore$

$T = mg(3 + \frac{x}{a}) - (3 + \frac{x}{a})m\frac{dv}{dt} = m(3 + \frac{x}{a})(g - \frac{dv}{dt})$ $\quad (ii)$

$\frac{d}{dx}(i)$: $(\pi + 7)a2v\frac{dv}{dx} = 2g(a + 2x)$

$\therefore \quad v\frac{dv}{dx} = \frac{(a+2x)g}{(\pi+7)a}$

$\therefore \quad \frac{dv}{dt} = \frac{(a+2x)g}{(\pi+7)a}$

Sub in (ii) : $T = m(3 + \frac{x}{a})(g - \frac{(a+2x)g}{(\pi+7)a})$

When $x = 2a$, $\quad T = 5mg(1 - \frac{5}{\pi+7}) = 5mg\frac{\pi+2}{\pi+7}$ (ANS)

ANSWER, Second Part, Third Method

$\frac{d}{dx}(i)$: $\quad (\pi + 7)a2v\frac{dv}{dx} = 2g(a + 2x)$

$\therefore \quad v\frac{dv}{dx} = \frac{(a+2x)g}{(\pi+7)a}$

When $x = 2a$, $\quad v\frac{dv}{dx} = \frac{5g}{\pi+7}$

Newton ↓, particle: $\quad mg - T_0 = m\frac{5g}{\pi+7}$

Newton ↓, element:

$T + mg\frac{\delta y}{a} - (T + \delta T) = m\frac{\delta y}{a}\frac{5g}{\pi+7}$

$\therefore \quad \frac{mg}{a}(1 - \frac{5}{\pi+7})\delta y = \delta T$

$\therefore \quad \int_0^y \frac{mg}{a}\frac{\pi+2}{\pi+7}\,dy = \int_{T_0}^T dT$

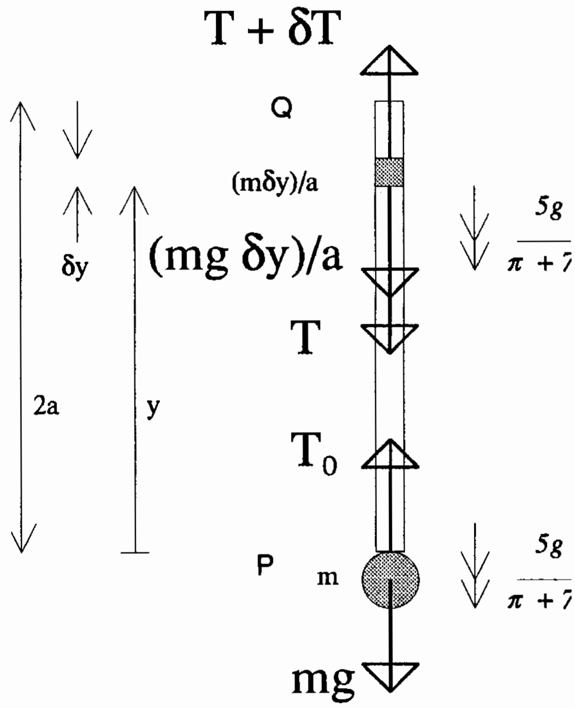
$\therefore \quad \frac{mg}{a}\frac{\pi+2}{\pi+7}y = T - T_0$

Sub for T_0 :

$\quad T = \frac{mg}{a}\frac{\pi+2}{\pi+7}(y + 1)$

When $y = 4a$:

$\quad T = 5mg\frac{\pi+2}{\pi+7}$ (ANS)



ANSWER, Second Part, Fourth Method

$\frac{d}{dx}(i)$: $\quad (\pi + 7)a2v\frac{dv}{dx} = 2g(a + 2x)$

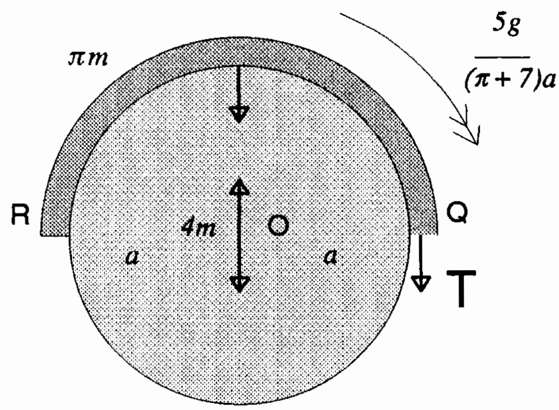
$\therefore \quad v\frac{dv}{dx} = \frac{(a+2x)g}{(\pi+7)a}$

When $x = 2a$, $\quad v\frac{dv}{dx} = \frac{5g}{\pi+7}$

Newton O↻ :

$\quad Ta = (\frac{1}{2}(4m)a^2 + \pi ma^2)\frac{5g}{(\pi+7)a}$

$\therefore \quad T = 5mg\frac{2+\pi}{\pi+7}$ (ANS)



ANSWER, Second Part, Fifth Method (as instructed)

Let momentum of the specified part of the system be ζ. Then

$$\zeta = (\tfrac{m}{a}(2a + x) + m)v = (3 + \tfrac{x}{a})mv$$

$$\therefore \quad \tfrac{d\zeta}{dt} = \tfrac{v}{a}mv + (3 + \tfrac{x}{a})m\tfrac{dv}{dt} \qquad (iii)$$

$$\tfrac{d}{dx}(i) : \quad (\pi + 7)a2v\tfrac{dv}{dx} = 2g(a + 2x)$$

$$\therefore \quad v\tfrac{dv}{dx} = \tfrac{(a+2x)g}{(\pi+7)a}$$

$$\therefore \quad \tfrac{dv}{dt} = \tfrac{(a+2x)g}{(\pi+7)a}$$

Sub in (iii), and for v^2 from (i):

$$\tfrac{d\zeta}{dt} = \tfrac{1}{a}m\tfrac{2g(ax+x^2)}{(\pi+7)a} + (3 + \tfrac{x}{a})m\tfrac{(a+2x)g}{(\pi+7)a}$$

When $x = 2a$,

$$\tfrac{d\zeta}{dt} = m\tfrac{2g6}{\pi+7} + 5m\tfrac{5g}{\pi+7} = \tfrac{37mg}{\pi+7}$$

Using the suggested piece of theory for the time when $x = 2a$:

$$mg + 4mg - T = \tfrac{37mg}{\pi+7}$$

$$\therefore \quad T = 5mg - \tfrac{37mg}{\pi+7} = \tfrac{5\pi-2}{\pi+7}mg, \qquad \textit{as desired.}$$

ANSWER, Second Part, Sixth Method (as instructed)

Let momentum of the specified part of the system be ζ. Then

$$\zeta = (\tfrac{m}{a}(2a + x) + m)v = (3 + \tfrac{x}{a})mv$$

$$\therefore \quad \zeta^2 = (3 + \tfrac{x}{a})^2 m^2 v^2$$

Sub for v^2 from (i):

$$\zeta^2 = (3 + \tfrac{x}{a})^2 m^2 \tfrac{2g(ax+x^2)}{(\pi+7)a}$$

$$\therefore \quad 2\zeta\tfrac{d\zeta}{dt} = 2(3 + \tfrac{x}{a})\tfrac{v}{a}m^2\tfrac{2g(ax+x^2)}{(\pi+7)a} + (3 + \tfrac{x}{a})^2 m^2 \tfrac{2g(av+2xv)}{(\pi+7)a}$$

Sub for ζ and cancel $2mv$:

$$(3 + \tfrac{x}{a})\tfrac{d\zeta}{dt} = (3 + \tfrac{x}{a})(\tfrac{m}{a}\tfrac{2g(ax+x^2)}{(\pi+7)a} + (3 + \tfrac{x}{a})m\tfrac{g(a+2x)}{(\pi+7)a})$$

When $x = 2a$,

$$\tfrac{d\zeta}{dt} = \tfrac{2mg6}{(\pi+7)} + 5m\tfrac{5g}{(\pi+7)} = \tfrac{37mg}{\pi+7}$$

Using the suggested piece of theory for the time when $x = 2a$:

$$mg + 4mg - T = \tfrac{37mg}{\pi+7}$$

$$\therefore \quad T = 5mg - \tfrac{37mg}{\pi+7} = \tfrac{5\pi-2}{\pi+7}mg, \qquad \textit{as desired.}$$