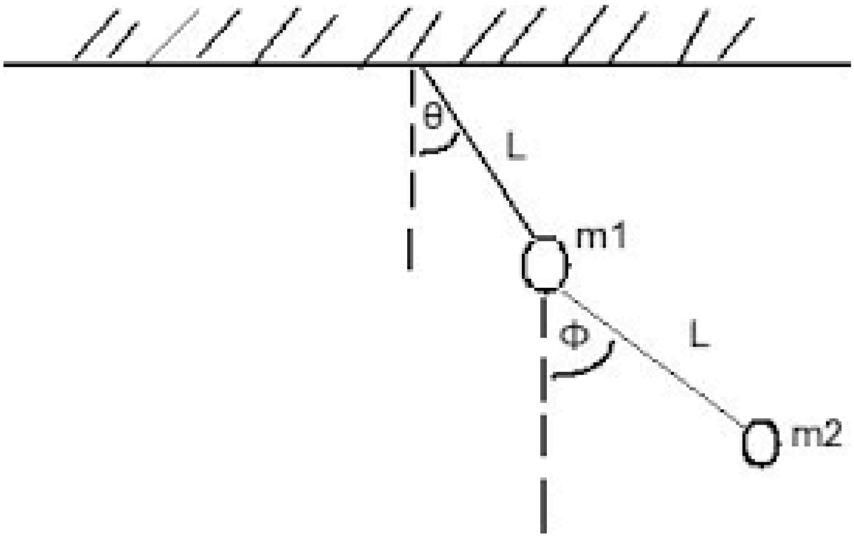
2.5.7 Double Pendulum



Consider a simple pendulum of length L with a bob of mass m_1, to which is attached a second simple pendulum of length L with a bob of mass m_2. The first pendulum can oscillate about the suspension point with angular coordinate θ_1, and the second pendulum can oscillate about the mass m_1 with angular coordinate θ_2.

The potential energy of this system is given by:

$$V = -m_1 gL\cos\theta_1 - m_2 gL(\cos\theta_1 + \cos\theta_2)$$

Expanding this in a small angle approximation $\cos\theta = 1 - \theta^2/2$:

$$V = V_0 + \frac{gL}{2}\left[(m_1 + m_2)\theta_1^2 + m_2\theta_2^2\right]$$

where V_0 is independent of the θ coordinates and will be ignored from now on. The remaining term represents the variation of V during oscillations.

The kinetic energy of the system is a bit more tricky. For the mass m_1 it is just:

$$T_1 = \frac{1}{2}I\omega^2 = \frac{1}{2}m_1 L^2\left(\frac{d\theta_1}{dt}\right)^2$$

For the mass m_2 it is simplest to work in cartesian coordinates x and y:

$$\frac{dx}{dt} = L\left[\cos\theta_1\left(\frac{d\theta_1}{dt}\right) + \cos\theta_2\left(\frac{d\theta_2}{dt}\right)\right]$$

$$\frac{dy}{dt} = -L\left[\sin\theta_1\left(\frac{d\theta_1}{dt}\right) + \sin\theta_2\left(\frac{d\theta_2}{dt}\right)\right]$$

$$T_2 = \frac{1}{2}m_2L^2\left[\left(\frac{d\theta_1}{dt}\right)^2 + \left(\frac{d\theta_2}{dt}\right)^2 + 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\left(\frac{d\theta_1}{dt}\right)\left(\frac{d\theta_2}{dt}\right)\right]$$

For small angles the coefficient in the angular coordinates can be approximated:

$$\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2) \approx 1$$

and the total kinetic energy is:

$$T = T_1 + T_2 = \frac{1}{2}L^2\left[(m_1 + m_2)\left(\frac{d\theta_1}{dt}\right)^2 + 2m_2\left(\frac{d\theta_1}{dt}\right)\left(\frac{d\theta_2}{dt}\right) + m_2\left(\frac{d\theta_2}{dt}\right)^2\right]$$

The kinetic and potential energies can be written in matrix form:

$$T = \frac{1}{2}L^2\left(\frac{d\theta_i}{dt}\right)^T M\left(\frac{d\theta_i}{dt}\right) \qquad M = \begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}$$

$$V = \frac{1}{2}gL\theta_i^T K\theta_i \qquad K = \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix}$$

Total energy conservation $dE/dt = 0$ leads to the equation:

$$M\frac{d^2\theta_i}{dt^2} + K\theta_i = 0$$

as already shown in an earlier section. We make a change of variables:

$$\alpha_i = \begin{pmatrix} 1/\mu & 0 \\ 0 & 1 \end{pmatrix}\theta_i$$

where $\mu = \sqrt{m_2/(m_1 + m_2)}$. With these coordinates, and taking out a factor m_2, the energy matrices become:

$$M' = \begin{pmatrix} 1 & \mu \\ \mu & 1 \end{pmatrix} \qquad K' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues of the motion can be found from the determinant:

$$|M' - \lambda K'| = \begin{vmatrix} 1-\lambda & \mu \\ \mu & 1-\lambda \end{vmatrix} = 0$$

The solutions are $\lambda_1 = 1 + \mu$ and $\lambda_2 = 1 - \mu$, and the corresponding eigenvectors are:

$$\alpha_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \alpha_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The two pendulums are in phase in the first solution and out of phase in the second solution. Putting back in the constants g, L, m_1 and m_2 in the appropriate places the frequencies of the normal modes are:

$$\omega_1 = \sqrt{\frac{g}{L(1+\mu)}} \qquad \omega_2 = \sqrt{\frac{g}{L(1-\mu)}}$$

and the normal coordinates are:

$$\theta_i = \frac{1}{L} \begin{pmatrix} \sqrt{1/(m_1+m_2)} & 0 \\ 0 & \sqrt{1/m_2} \end{pmatrix} \alpha_i$$

When $m_2 \ll m_1$, $\mu \to 0$ the solutions become degenerate and represent the simple pendulum with m_1 and length L. When $m_2 \gg m_1$, $\mu \to 1$ the higher mode $\omega_2 \to \infty$, and the lower mode represents a simple pendulum with m_2 and length $2L$.