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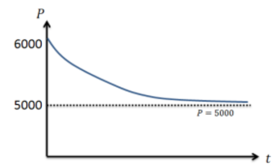
C3 Cheat Sheet

Chapter	Usual types of questions	Tips	What can go ugly
1 – Algebraic Fractions	• Almost always adding or subtracting fractions. • Simplifying top heavy fraction using algebraic division. • Simplifying fractions by first factorising numerator and denominator, where possible.	• Factorise everything in each fraction first. e.g. If denominators $(2x+1)(x-3)$ and x^2-9, common denominator will be $(2x+1)(x+3)(x-3)$ • Otherwise, just a case of practice! • If adding/subtracting a constant, turn into a fraction. $$\frac{x+1}{x+2}-1 \rightarrow \frac{x+1}{x+2}-\frac{x+2}{x+2}$$ • Suppose we were asked to turn $\frac{x^2+1}{x-1}$ into a mixed number. By algebraic division we find the quotient is $x+1$ and the remainder 2. Remember we can express the result as $Q(x)+\frac{R(x)}{D(x)}$ where $Q(x)$ is the quotient, $R(x)$ the remainder and $D(x)$ the divisor, just as $16 \div 3 = 5 + \frac{1}{3}$ where 5 was the quotient and 1 the remainder. Thus $\frac{x^2+1}{x-1} \equiv x+1+\frac{2}{x-1}$ • In general, to simplify 'fractions within fractions', multiply top and bottom of the outer fraction by the denominator of the inner fraction, e.g.: $$\frac{\frac{a}{bc}+1}{c}=\frac{a+bc}{bc^2}$$	• Blindly multiplying the two denominators when there might be a common factor, e.g. $(x+3)(x-3)$ and $(x+3)$ should become $(x+3)(x-3)$ rather than $(x+3)^2(x-3)$. • Classic sign errors when subtracting a fraction. $$\frac{3x}{x-2}-\frac{1}{x+1} \rightarrow \frac{3x(x+1)-(x-2)}{(x-2)(x+1)}$$ Note use of brackets around $(x-2)$ ensures -2 becomes +2.

2 – Functions	• Specifying range or domain of function.
	• Finding inverse of function.
	• Finding specific output of function, e.g. $f(3)$ or $fg(4)$
	• Finding specific output using graph only (without explicit definition of function)
	• Finding composite function.
	• Sketching original function and inverse function on same axis (i.e. reflection in $y=x$)
	• Be able to find $f(x^2)$ or $f(2x)$ for example, when the function $f(x)$ is known.
	• "If $f(x)=x^2-3$ find all values of x for which $f(x)=f^{-1}(x)$"
		• You should know and understand why $f^{-1}f(x)=x$
		• I avoid getting domain and range mixed up by thinking "ddrrr" with a silly voice. i.e. <u>**D**</u>omain first (possible inputs) followed by <u>**R**</u>ange (possible outputs)
		• Learn the domains and ranges of each of the 'common functions' ($\frac{1}{x}, \sin(x), \sqrt{x}, e^x, \ln x, (x+a)^2+b, \ldots$). Be careful about > vs ≥
		• Domains can be restricted for 2 main reasons:
		  ○ The denominator of a fraction can't be 0, so domain of $f(x)=\frac{1}{2x+1}$ is $x \neq -\frac{1}{2}$. Use ≠ symbol.
		  ○ You can't square root a negative number, so range of $f(x)=\sqrt{x}$ is $x \geq 0$. Similarly you can't log 0 or negative numbers so domain $x>0$ (notice it's strict)
		• Range can be restricted for 3 main reasons:
		  ○ The domain was restricted. e.g. If $f(x)=x^2$ and domain is set to $3<x\leq 4$, then range is $9<f(x)\leq 16$. Notice strictness/non-strictness of bounds.
		  ○ Asymptotes. For reciprocals a division can never yield 0
			• Having a lack of care in domains/ranges with the strictness/nonstrictness of the bound. For $f(x)=e^x$, range is $f(x)>0$ not $f(x)\geq 0$. Similarly range for quadratics are non-strict because min/max point is included.
			• Putting the range of a function in terms of x instead of say the correct $f(x)$. Similarly for the range of an inverse, if the domain of the original function was say $x>2$, then the range of the inverse is $f^{-1}(x)>2$ (i.e. even though the inequality is effectively the same, we're referring to the output of f^{-1} now so need to use $f^{-1}(x)$ rather than x)
			• When finding $fg(x)$, accidentally doing f first and then g.
			• If $f(x)=\ln x$, then you should recognise that $f(x^2)=\ln(x^2)$, <u>NOT</u> $(\ln(x))^2$, which would suggest you don't quite fully understand how

		(unless numerator is 0), thus range of $f(x) = \frac{1}{x+3}$ is $f(x) \neq 0$. Use sketch! For exponential functions, output is (strictly) positive: $f(x) > 0$ ◦ Min/max value of a quadratic (or any polynomial whose highest power is even). Note bound is non-strict as min/max value included. Range of $f(x) = (x+a)^2 + b$ is $f(x) \geq b$ • For composite functions, if given say $fg(x)$, write as $f(g(x))$ then substitute $g(x)$ with its definition so you have $f(some\ expression)$. You're less likely to go wrong. • Remember that domain is specified in terms of x and the range in terms of $f(x)$ (or $g(x)$ or otherwise). • Remember that the domain and range is swapped for the inverse function. • If the domain is unrestricted, you should know to write $x \in \mathbb{R}$, where the ∈ means "is a member of" and ℝ is "the set of real number". Similarly an unrestricted range is $f(x) \in \mathbb{R}$. • If asked "Why does the function not have an inverse", answer with "It is a many-to-one function." • "If $f(x) = x^2 - 3$ find all values of x for which $f(x) = f^{-1}(x)$". If $f(x) = f^{-1}(x)$, this only occurs when the input is equal to the output, i.e. $f(x) = x$. Thus solve $x = x^2 - 3$. • Sometimes you never actually have the explicit function, but are provided with a graph. Suppose $(3,5)$ and $(5,0)$ were points on the graph. Then $f(3) = 5, f^{-1}(5) = 3$, and $ff(3) = f(f(3)) = f(5) = 0$. For harder questions you must have a combination of a sketch (say for f) and an explicitly given one (say g). If say $fg(x) = 3$ then note that $x = g^{-1}f^{-1}(3)$: we'd first find $f^{-1}(3)$ by using the line graph backwards, and sub this value into your function for g^{-1}, or set $g(x)$ equal to this value and solve.	functions work.

3 – Exponential and Log Functions	• Be able to draw the graphs of $y = \ln x$ and $y = e^x$. • Sketch more complicated exponential graphs, e.g. $y = 100 + 50e^{-x}$ • Solve equations involving ln and e (see right). • Solve equations which are quadratic in terms of e^x (see right). • Find the 'long term' value of an exponential graph.	• The log function exists to provide an inverse of the exponential function, e.g. if we use $f(x) = 2^x$ then we could obtain original value using $f^{-1}(x) = \log_2 x$. • C2 laws of logs you are expected to know and use in C3/C4: $\log a + \log b = \log(ab)$ $\log a - \log b = \log\left(\frac{a}{b}\right)$ $\log a^b = b \log a$ • Combine ln's first in the same way as you would in C2. e.g. $\ln 3x + \ln x^2 = y$ becomes $\ln 3x^3 = y$ which in turn becomes $3x^3 = e^y$ • For equations of the following form, just 'ln' both sides first and use laws of logs to split up the LHS: $2^x e^x = 5$ $\ln 2^x\, e^x = \ln 5$ $\ln 2^x + \ln e^x = \ln 5$ $x \ln 2 + x = \ln 5 \qquad x(\ln 2 + 1) = \ln 5$ $x = \frac{\ln 5}{\ln 2 + 1}$ • Some questions are 'quadratics in disguise': $e^x + 3e^{-x} = 4$ Since $e^{-x} = \frac{1}{e^x}$, this suggest we multiply everything by e^x, which gives us: $(e^x)^2 + 3 = 4e^x$. Rearranging: $(e^x)^2 - 4e^x + 3 = 0$ Then we could make the substitution $y = e^x$ and factorise, or just factorise immediately to get $(e^x - 1)(e^x - 3) = 0$ Thus $e^x = 1 \; or \; e^x = 3$, thus $x = 0 \; or \; x = \ln 3$. • If you'd managed to factorise an expression to say $e^x(x+1) = 0$, remember that e^x can't be 0 (as the range of exponential functions is $e^x > 0$) • Suppose the population is given by $P = 5000 + 1000e^{-3t}$ where t is the number of years. What is the initial population? What is the long-term population? If $t = 0, P = 5000 + 1000e^0 = 5000 + 1000 = 6000$ As $t \to \infty$, $e^{-3t} = \frac{1}{e^{3t}}$ will tends towards 0. Thus $P = 5000$ • To draw the graph of $P = 5000 + 1000e^{-3t}$, either do as a graph transform from $P = e^{-t}$ (i.e. shift up 5000, ensuring you get y-intercept correct), or consider initial value and long term value as above. Ensure you draw asymptote!  	• The worst algebraic error you can make is going from say $\ln(x) + \ln(2) = y$ to: $x + 2 = e^y$. Applying e to both sides doesn't apply it individually to each thing in a sum. It's equivalent to the misconception that $\sqrt{a^2 + b^2} \equiv a + b$. To 'undo' a ln, you have to isolate a single ln on one side of the equation with nothing being added or subtracted from it (as per the example on the left) • Misremembering C2 laws of logs. • Thinking $e^0 = 0$ particularly in applied questions (e.g. population growth).

4 – Numerical methods	• Rearranging an equation into the form $x = g(x)$, i.e. where x appears on both sides of the equation, but x appears in isolation on one side. • Using a recurrence to get successive approximations. • Justifying why a value is correct to a given number of decimal places. • Justifying why a root lies in a given range. Similarly, justifying why a turning point lies in a given range (see right).	• When asked to show that you can rearrange an equation to the form $x = g(x)$, use the structure of the target equation to yield clues about how to rearrange (since there are multiple ways of isolating x, some which will lead to dead ends) • Exploit the ANS key on your calculator to get successive approximations. Put your x_0 value into the value then immediately press =. Then write an expression in terms of ANS and spam your = key. • Remember the golden words "change of sign" when justifying why a root is in a range. • If asked to justify why a max/min point is in the range, note that the <u>gradient</u> goes from positive to negative (or vice versa) and thus there is a change of sign. • "Show that the solution to $f(x) = g(x)$" lies in the range [3,4]." Rewrite as $f(x) - g(x) = 0$, so that we can then evaluate $f(3) - g(3)$ and $f(4) - g(4)$ and show there's a change in sign.	• Using radians instead of degrees, or vice versa, for recurrences involving trig functions. If your values don't gradually converge (i.e. approach) to a particular value when you spam the = key, something probably went wrong. • Reaching a dead end when trying to rearrange the equation to give a certain form.
5 – Transforming graphs of functions	• Using an existing function $y = f(x)$ to sketch $y = f(\|x\|)$ and/or $y = \|f(x)\|$ • As at GCSE/C1, be able to sketch a variety of functions by considering the transformation involved, e.g. $y = \frac{1}{2}\cos\left(x + \frac{\pi}{4}\right)$ • Solve equations of the form $\|f(x)\| = g(x)$ • Note that the specification explicitly excludes multiple transformations to the input of a function, i.e. you will NOT see $y = f(ax + b)$	• The way to remember $f(\|x\|)$ vs $\|f(x)\|$ is remember that changes inside the function brackets affects the x values and changes outside affect the y values. Thus in $y = f(\|x\|)$, any negative x is made positive before being inputted into the function. Thus we copy and reflect the graph from the right side of the y-axis (and discard anything that was on the left). • For curved graphs, at the point of reflection, the line should be sharp (i.e. a sudden change in direction) not smooth. • For questions such as say "Solve $\|3x - 2\| = x + 4$" then for any expression involving a |..|, have a separate equation where it is positive or negative. i.e. $3x - 2 = x + 4 \rightarrow x = 3$ $-3x + 2 = x + 4 \rightarrow x = -\frac{1}{2}$ However you MUST check both solutions satisfy the original equation: $\|3(3) - 2\| = 7 \quad 3 + 4 = 7$ $\left\|3\left(-\frac{1}{2}\right) - 2\right\| = \frac{7}{2} \quad -\frac{1}{2} + 4 = \frac{7}{2}$ Thus both solutions in this case are valid.	• Forgetting to add key coordinates to your diagrams, e.g. intercepts with the axis, turning points, etc. • Not checking your solutions are valid when solving equations of the form $\|f(x)\| = g(x)$ or similar. • If considering $\|2x - 5\|$ in solving an equation, it would be a mistake to consider $2x - 5$ and $2x + 5$. You actually want $2x - 5$ and $-2x + 5$. i.e. Negate the whole expression, don't just make each negative term positive!

6/7 - Trigonometry	Questions mainly in two flavours: • Solvey questions. • Provey questions. Either type may involve use of double angle/angle sum formulae, or identities $1+\tan^2\theta \equiv \sec^2\theta$ or $1+\cot^2\theta \equiv cosec^2\theta$ Be able to express $a\cos\theta + b\sin\theta$ in the form $Rcos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$	• To remember which way the tan and sec go in $1+\tan^2\theta \equiv \sec^2\theta$, I remember the queen coming back from holiday, saying "one is tanned". Then slap a 'co' on the front of the tan and sec to get the second identity. • For proof questions, if you have a mixture of say $2x$ and x, it's generally easiest to put everything in terms of x first using double angle formulae. E.g.: $\cos 2x \cos x + \sin 2x \sin x \equiv \cos x$ • Similarly for proof questions, when you have a mix of sin, cos, tan, sec and the like, whenever you're stuck, just write EVERYTHING in terms of sin and cos. • In proof questions, when you have fractions being added/subtracted, combine into a single fraction. Things usually end up cancelling. • Remember the '5 golden rules of trig angles', and you won't need those silly 'CAST diagrams': 1. $\mathbf{\sin(x) = \sin(180 - x)}$ **2.** $\mathbf{\cos(x) = \cos(360 - x)}$ **3. $\boldsymbol{sin/cos}$ repeat every 360.** **4. $\boldsymbol{tan}$ repeats every 180.** **5.** $\mathbf{\sin(x) = \cos(90 - x)}$ (lack of knowledge of this one cost students dearly in the June 2013 paper!) • As with C2 questions, when you have a combination of say $\tan^2 x$ and $sec\, x$, then change the squared term using the identity, so that you end up with a quadratic equation in terms of one trig function. • As per C2, if you're given a range for your solutions, then rewrite the range as appropriate. E.g. If $0 < x < \pi$ and you had $\sin(2x) = \frac{1}{2}$, then $0 < 2x < 2\pi$. This ensures you don't lose solutions. • To solve $\cot(2x) = 0$, we'd usually reciprocate both sides so that $\cot(2x)$ becomes $\tan(2x)$, but we can't do $\frac{1}{0}$! We can see this would happen at asymptotes for tan, thus $2x = \frac{\pi}{2}$	• Forgetting solutions in solvey questions. • Forgetting the $\pm$ when say dealing with $\tan^2 x = 2$ (and thus losing solutions). • Dividing both sides of a equation by say $\sin x$, rather than moving everything to one side and factorising (as you again would lose solutions). • Getting to a dead end in proof questions (as per advice, combine any fractions into a single one, and write everything in terms of sin and cos if completely stuck). • I've seen some students try to skip steps with $a\cos\theta + b\sin\theta$ questions, and end up making errors particularly in working out α. Ensure you show the expansion of $Rcos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$, compare coefficients, and divide $Rsin\ \alpha$ and $R\cos\alpha$ the correct way to get $\tan\alpha$.

8 - Differentiation	• Use of one or more of product rule, quotient rule, chain rule. • Differentiating e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$. • $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$ • Be able to simplify a more complicated expression via factorisation (see tips).	• Don't use the quotient rule if the numerator is a constant – it's simpler to re-express as a product: $$y = \frac{3}{(2x-5)^4} = 3(2x-5)^{-4}$$ $$\frac{dy}{dx} = 3(-4)(2x-5)^{-5} \times 2$$ • To remember the signage of differentiating and integrating sin and cos, I visualise sin above cos, and that differentiating is 'going down' and integrating 'going up'. If you have to go 'the wrong way' (e.g. integrating sin, but this is at the top) then the sign changes. • For nested functions, we obviously use chain rule. I use the "bla method": "Differentiating the outer function with respect to bla, then times by bla differentiated". e.g. with e^{x^2+3}: "e to the bla" differentiated is "e to the bla" (i.e. e^{x^2+3}) and "bla differentiated" is $2x$. • If x is expressed in terms of y and you need to find $\frac{dy}{dx}$, then sometimes it is not convenient to make y the subject first: instead find $\frac{dx}{dy}$ and then take the reciprocal to get $\frac{dy}{dx}$. Generally $\frac{dx}{dy}$ leads to an expression in terms of y. Use trig identities or otherwise to get back in terms of x. For example: $$x = \sin y$$ $$\frac{dx}{dy} = \cos y$$ $$\frac{dy}{dx} = \frac{1}{\cos y}$$ But we need $\sin y$ to get in terms of x. Since $\sin^2 y + \cos^2 y = 1$: $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-x^2}}$$ • When differentiating a trig function to some power, write first putting the power outside a bracket, so that it's clearer you should be using chain rule: $$y = \sin^3 2x = (\sin 2x)^3$$ $$\frac{dy}{dx} = 3(\sin 2x)^2 \times 2\cos 2x$$ • You will often have to simplify a differentiated expression. Remember that when factorising things with indices, <u>factor out the smallest power</u>, and factor out any fraction using the lowest	• Substituting into the quotient rule wrong. E.g. Doing $u\frac{dv}{dx} - v\frac{du}{dx}$ instead of the correct $v\frac{du}{dx} - u\frac{dv}{dx}$ • Doing something horrid like this: $\frac{d}{dx}(e^{3x}) = 3x\, e^{3x}$ • Or even worse: $\frac{d}{dx}(e^{3x}) = 3x\, e^{3x-1}$ • In general, forgetting to use the chain rule. If there's some nested linear expression, make sure you always multiply by the coefficient of x: $$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(3x)) = -3\sin 3x$$ • Getting the signs wrong when differentiating (or integrate) sin and cos, particularly when the chain rule is involved. e.g. Incorrectly differentiating $\sin(2-x)$ to $\cos(2-x)$ rather than $-\cos(2-x)$ due to chain rule. • Note that $\frac{1}{2(x-1)^3}$ is equal to $\frac{1}{2}(x-1)^{-3}$, NOT $2(x-1)^{-3}$! • If using the product rule with more complex expressions (e.g. with considerable use of chain rule, particularly when there are lots of negatives floating about), you are advised to work out $\frac{du}{dx}$ and $\frac{dv}{dx}$ separately first rather than write all in one go.

		common multiple: $y = \frac{1}{6x(x-1)^{\frac{1}{2}}} = \frac{1}{6}x^{-1}(x-1)^{-\frac{1}{2}}$ $\frac{dy}{dx} = \frac{1}{6}x^{-1}\left(-\frac{1}{2}\right)(x-1)^{-\frac{3}{2}} + \frac{1}{6}(-1)x^{-2}(x-1)^{-\frac{1}{2}}$ $= -\frac{1}{12}x^{-1}(x-1)^{-\frac{3}{2}} - \frac{1}{6}x^{-2}(x-1)^{-\frac{1}{2}}$ $= \frac{1}{12}x^{-2}(x-1)^{-\frac{3}{2}}[-x-2(x-1)] \quad = -\frac{1}{12}x^{-2}(x-1)^{-\frac{3}{2}}(2-3x)$ • If the "tangent is parallel to the y-axis", then its gradient is infinite: this would only happen because of a division by 0 in the original equation. e.g. If $y = \frac{1}{x+3}$, then the 'tangent' is effectively the asymptote, which has equation $x = -3$.	