

2016 STEP I Q2

Let $y = (ax^2 + bx + c) \ln(x + \sqrt{1+x^2}) + (dx + e)\sqrt{1+x^2}$

$$\Rightarrow \frac{dy}{dx} = (2ax + b) \ln(x + \sqrt{1+x^2}) + (ax^2 + bx + c) \left(\frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \right) + d\sqrt{1+x^2} + \frac{dx^2 + ex}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{dy}{dx} = (2ax + b) \ln(x + \sqrt{1+x^2}) + \frac{(a+d)x^2 + (b+e)x + c}{\sqrt{1+x^2}} + d\sqrt{1+x^2}$$

Part (i)

Consider when $a = 0$, $b = 1$, $c = 0$, $d = 0$, $e = -1$

$$y = x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} \quad \text{and} \quad \frac{dy}{dx} = \ln(x + \sqrt{1+x^2})$$

$$\text{Hence } \int \ln(x + \sqrt{1+x^2}) dx = x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + k$$

Part (ii)

Consider when $a = 0$, $b = 0$, $c = \frac{1}{2}$, $d = \frac{1}{2}$, $e = 0$

$$y = \frac{1}{2} (\ln(x + \sqrt{1+x^2}) + x\sqrt{1+x^2}) \quad \text{and} \quad \frac{dy}{dx} = \sqrt{1+x^2}$$

$$\text{Hence } \int \sqrt{1+x^2} dx = \frac{1}{2} (\ln(x + \sqrt{1+x^2}) + x\sqrt{1+x^2}) + k$$

Part (iii)

Consider when $a = \frac{1}{2}$, $b = 0$, $c = \frac{1}{2}$, $d = 0$, $e = 0$

$$y = \frac{1}{2} (x^2 + 1) \ln(x + \sqrt{1+x^2}) \quad \text{and} \quad \frac{dy}{dx} = x \ln(x + \sqrt{1+x^2}) + \frac{1}{2} \sqrt{1+x^2}$$

$$\Rightarrow \int \left(x \ln(x + \sqrt{1+x^2}) + \frac{1}{2} \sqrt{1+x^2} \right) dx = \frac{1}{2} (x^2 + 1) \ln(x + \sqrt{1+x^2})$$

$$\Rightarrow \int x \ln(x + \sqrt{1+x^2}) dx = \frac{1}{2} (x^2 + 1) \ln(x + \sqrt{1+x^2}) - \frac{1}{2} \int \sqrt{1+x^2} dx$$

$$\Rightarrow \int x \ln(x + \sqrt{1+x^2}) dx = \frac{1}{4} (2x^2 + 1) \ln(x + \sqrt{1+x^2}) - \frac{1}{4} x\sqrt{1+x^2} + k$$