(87) To prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. (*)

LHS $= \int_0^a f(x)\,dx$

[Let $y = a-x$; $x = a-y$
$dx = -dy$
Limits: $(0, a) \to (a, 0)$]

$= \int_a^0 -f(a-y)\,dy$

$= \int_0^a f(a-y)\,dy$

[By the fundamental theorem of calculus $\int_\alpha^\beta f(t)\,dt = \int_\alpha^\beta f(u)\,du$]

$= \int_0^a f(a-x)\,dx$

$=$ RHS Hence Proved.

Scanned by CamScanner

(i) Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x} dx$ ..... (i)

[applying *)

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx \quad \text{.... (ii)}$$

Adding (i) & (ii)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x + \sin x} dx$$

$$\therefore 2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

(iii) Let $J = \int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\sin(x)+\cos(x)} \, dx.$

[Applying (*)]

$$= \int_0^{\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{4}-x\right)+\cos\left(\frac{\pi}{4}-x\right)} \, dx.$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x}{\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x} \, dx.$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{2\cos x} \, dx.$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 - \tan x) \, dx$$

$$= \frac{1}{2}\left(x - \ln \sec x\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \ln\sqrt{2}\right)$$

(iii) Let $K = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\, dx$

[Applying *]

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan x+1-\tan x\right) dx$$

$$- \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\, dx.$$

$$\Rightarrow 2K = x \ln 2 \Big|_0^{\frac{\pi}{4}}$$

$$\Rightarrow K = \frac{\pi}{8} \ln 2.$$

iv) Let $L = \int_0^{\frac{\pi}{4}} \frac{x}{\cos x(\cos x + \sin x)} dx.$

[Applying *]

$$= \int_0^{\frac{\pi}{4}} \frac{\frac{\pi}{4} - x}{\cos\left(\frac{\pi}{4} - x\right)\left(\cos\left(\frac{\pi}{4} - x\right) + \sin\left(\frac{\pi}{4} - x\right)\right)} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\frac{\pi}{4} - x}{\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)\left(\sqrt{2}\cos x\right)} dx$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{4}} \frac{1}{\cos x(\cos x + \sin x)} dx - \int_0^{\frac{\pi}{4}} \frac{x}{\cos x(\cos x + \sin x)} dx$$

$$2L = \frac{\pi}{4}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + \cos x \sin x} dx$$

$$I = \frac{\pi}{8} \int_0^{\frac{\pi}{4}} \frac{1}{1 + \frac{\cos 2x}{2} + \frac{\sin 2x}{2}} \, dx$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{4}} \frac{1}{1 + \frac{1 - \tan^2 x}{1 + \tan^2 x} + \frac{2\tan x}{1 + \tan^2 x}} \, dx$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{4}} \frac{1 + \tan^2 x}{2 + 2\tan x} \, dx$$

$$= \frac{\pi}{8} \ln(1 + \tan x) \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} \ln 2$$