STEP II 2016 Solutions

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1 The curve C_1 has parametric equations $x = t^2, y = t^3$, where $-\infty < t < \infty$. Let O denote the point $(0, 0)$. The points P and Q on C_1 are such that $\angle POQ$ is a right angle. Show that the tangents to C_1 at P and Q intersect on the curve C_2 with equation $4y^2 = 3x - 1$.

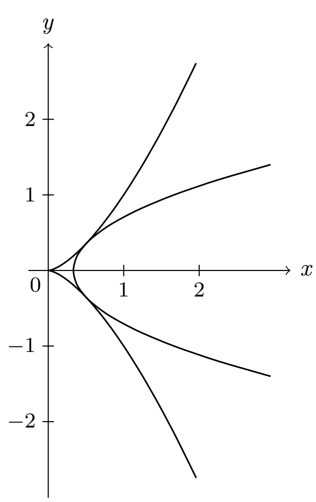
Determine whether C_1 and C_2 meet, and sketch the two curves on the same axes.

Solution by riquix.

Let $t = p$ at P and $t = q$ at Q. Then P is (p^2, p^3) and Q is (q^2, q^3).
Then line OP has gradient $\dfrac{p^3 - 0}{p^2 - 0} = p$, and line OQ has gradient q.
$\angle POQ$ is a right angle $\iff$ OP is perpendicular to OQ $\iff$ $pq = -1$ $\iff$ $q = -\dfrac{1}{p}$.
The gradient of C_1 at the point with parameter t is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t} \div \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{3t^2}{2t} = \dfrac{3}{2}t$.
So the tangent to C_1 at P has equation $y = \dfrac{3}{2}p(x - p^2) + p^3$,
and the tangent at Q has equation $y = \dfrac{3}{2}q(x - q^2) + q^3 = -\dfrac{3}{2p}(x - \dfrac{1}{p^2}) - \dfrac{1}{p^3}$.
$\implies$ At the point of intersection between the tangents, $\dfrac{3}{2}p(x - p^2) + p^3 = -\dfrac{3}{2p}(x - \dfrac{1}{p^2}) - \dfrac{1}{p^3}$,
which gives $x = \dfrac{p^6 + 1}{3p^2(p^2 + 1)}$.
Substituting this into either tangent equation gives $y = \dfrac{1 - p^2}{2p}$.
These give $3x - 1 = \dfrac{p^6 - p^4 - p^2 + 1}{p^2(p^2 + 1)}$ and $4y^2 = \dfrac{p^4 - 2p^2 + p^4}{p^2} = \dfrac{p^6 - p^4 - p^2 + 1}{p^2(p^2 + 1)}$,
so $4y^2 = 3x - 1$ at the point of intersection of the two tangents, meaning it lies on the curve C_2.

For the two curves to meet, $4(t^3)^2 = 3(t^2) - 1 \iff 4t^6 - 3t^2 + 1 = 0 \iff (t^2 + 1)(4t^4 - 4t^2 + 1) = 0$
$\iff (2t^2 - 1)^2 = 0 \iff t = \pm\dfrac{1}{\sqrt{2}}$.

$$\boxed{\text{Hence the two curves do meet, at the points } \left(\frac{1}{2},\ \pm\frac{1}{2\sqrt{2}}\right).}$$



2 Use the factor theorem to show that $a+b-c$ is a factor of

$$(a+b+c)^3-6(a+b+c)(a^2+b^2+c^2)+8(a^3+b^3+c^3). \qquad (*)$$

Hence factorise $(*)$ completely.

(i) Use the result above to solve the equation

$$(x+1)^3-3(x+1)(2x^2+5)+2(4x^3+13)=0.$$

(ii) By setting $d+e=c$, or otherwise, show that $(a+b-d-e)$ is a factor of

$$(a+b+d+e)^3-6(a+b+d+e)(a^2+b^2+d^2+e^2)+8(a^3+b^3+d^3+e^3)$$

and factorise this expression completely.

Hence solve the equation

$$(x+6)^3-6(x+6)(x^2+14)+8(x^3+36)=0.$$

Solution by Hauss.

Let $\mathrm{f}(a,b,c)=(a+b+c)^3-6(a+b+c)(a^2+b^2+c^2)+8(a^3+b^3+c^3)$.

$\mathrm{f}(a,b,a+b)=0$, as shown by some algebraic manipulation after substituting in $a+b$ for c.

$\implies (a+b-c)$ is a factor of $f(a,b,c)$.

$\mathrm{f}(a,b,c)$ is symmetric in $a,b,c,$ so $(a-b+c)$ and $(a-b-c)$ are also factors.

By consideration of the term with the highest power of a, or by wasting lots of time on algebra, we see that $\boxed{\mathrm{f}(a,b,c)=3(a+b-c)(a-b+c)(a-b-c)}$.

(i) $(x+1)^3-3(x+1)(2x^2+5)+2(4x^3+13)=(x+1)^3-6(x+1)\left(x^2+\frac{5}{2}\right)+8\left(x^3+\frac{13}{4}\right)$

$=\mathrm{f}\left(x,\frac{3}{2},-\frac{1}{2}\right)=3(x-2)(x-1)(x+2)=0 \implies \boxed{x=1,\ x=2,\ x=-2}$.

(ii) Let $\mathrm{g}(a,b,d,e)=(a+b+d+e)^3-6(a+b+d+e)(a^2+b^2+d^2+e^2)+8(a^3+b^3+d^3+e^3)$

Using $d+e=c$ gives $\mathrm{g}(a,b,d,e)=\mathrm{f}(a,b,c)+12(a+b+c)de-24cde$

$\mathrm{f}(a,b,c)$ has a factor $(a+b-c)=(a+b-d-e)$, and when $a+b=c, 12(a+b+c)de-24cde=0$

$\implies 12(a+b+c)de-24cde$ has a factor $(a+b-c)=(a+b-d-e) \implies \mathrm{g}(a,b,d,e)$ has a factor $(a+b-d-e)$.

As $\mathrm{g}(a,b,d,e)$ is symmetric in $a,b,d,$ and $e, \mathrm{g}(a,b,d,e)$ has factors $(a-b+d-e)$ and $(a-b-d+e)$.

By consideration of the term with the highest power of a, or by wasting lots more time on algebra, we see that $\boxed{\mathrm{g}(a,b,d,e)=3(a+b-d-e)(a-b+d-e)(a-b-d+e)}$.

Finally, $\mathrm{g}(x,1,2,3)=(x+6)^3-6(x+6)(x^2+14)+8(x^3+36)=3x(x-2)(x-4) \implies \boxed{x=0,\ x=2,\ x=4}$.

3 For each non-negative integer n, the polynomial f_n is defined by

$$\mathrm{f}_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

(i) Show that $\mathrm{f}'_n(x) = \mathrm{f}_{n-1}(x)$ (for $n \geqslant 1$).

(ii) Show that, if a is a real root of the equation

$$\mathrm{f}_n(x) = 0, \qquad (*)$$

then $a < 0$.

(iii) Let a and b be distinct real roots of $(*)$, for $n \geqslant 2$. Show that $\mathrm{f}'_n(a)\,\mathrm{f}'_n(b) > 0$ and use a sketch to deduce that $f_n(c) = 0$ for some number c between a and b.

Deduce that $(*)$ has at most one real root. How many real roots does $(*)$ have if n is odd? How many real roots does $(*)$ have if n is even?

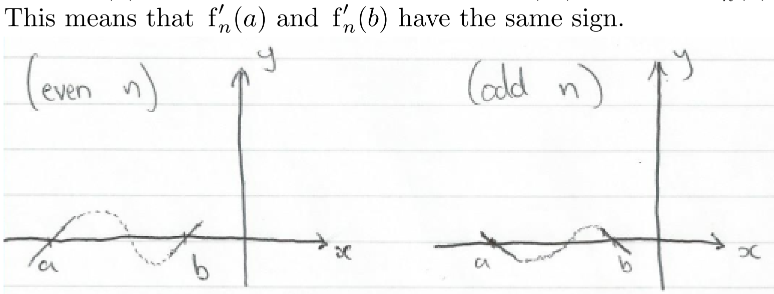
Solution by StrangeBanana.

(i) $\mathrm{f}_n(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$

$\implies \mathrm{f}'_n(x) = 0 + 1 + \dfrac{2x}{2!} + \dfrac{3x^2}{3!} + \cdots + \dfrac{nx^{n-1}}{n!} = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^{n-1}}{(n-1)!} = \mathrm{f}_{n-1}(x)$

(ii) If $\mathrm{f}_n(a) = 0$ for some $a \geqslant 0,$ then $1 + a + \dfrac{a^2}{2!} + \dfrac{a^3}{3!} + \cdots + \dfrac{a^n}{n!} = 0,$ but the LHS is a sum of non-negative terms (including at least one positive term), and hence greater than 0.
This is therefore a contradiction, so any root a must satisfy $a < 0$.

(iii) With two roots a and $b,$ taking $a < b,$ we have $\mathrm{f}'_n(a) = \mathrm{f}_{n-1}(a) = \mathrm{f}_n(a) - \dfrac{a^n}{n!} = -\dfrac{a^n}{n!},$
and similarly, $\mathrm{f}'_n(b) = -\dfrac{b^n}{n!},$ so $\mathrm{f}'_n(a)\,\mathrm{f}'_n(b) = \dfrac{(ab)^n}{(n!)^2}.$
By part (ii), we have $a, b < 0$ so $ab > 0 \implies (ab)^n > 0 \implies \mathrm{f}'_n(a)\,\mathrm{f}'_n(b) > 0,$ as required.
This means that $\mathrm{f}'_n(a)$ and $\mathrm{f}'_n(b)$ have the same sign.



Clearly, if the gradient is positive (or negative) at both roots, the curve must intersect the x axis somewhere between the two to maintain continuity. But this means that between any two roots of $\mathrm{f}_n(x) = 0$ is another distinct root, so we can find infinitely many roots.
As f_n is an n-degree polynomial, it can only have up to n real roots, so assuming that we have two distinct roots causes a contradiction. Hence, $\mathrm{f}_n(x) = 0$ has at most 1 real root.
This root cannot be repeated, as for any root $x \neq 0$ (as $\mathrm{f}_n(0) \equiv 1$), $\mathrm{f}'_n(x) = -\dfrac{x^n}{n!} \neq 0.$
To have real coefficients, the function must have an even number of non-real roots, so
$\boxed{\text{odd } n \implies 1 \text{ real root, and even } n \implies 0 \text{ real roots}}.$

4 Let

$$y = \frac{x^2 + x\sin\theta + 1}{x^2 + x\cos\theta + 1}.$$

(i) Given that x is real, show that

$$(y\cos\theta - \sin\theta)^2 \geqslant 4(y-1)^2.$$

Deduce that

$$y^2 + 1 \geqslant 4(y-1)^2,$$

and hence that

$$\frac{4-\sqrt{7}}{3} \leqslant y \leqslant \frac{4+\sqrt{7}}{3}.$$

(ii) In the case $y = \dfrac{4+\sqrt{7}}{3}$, show that

$$\sqrt{y^2+1} = 2(y-1)$$

and find the corresponding values of x and $\tan\theta$.

Solution by StrangeBanana.

(**i**) Rearrange the equation for y to a quadratic in x :
$(y-1)x^2 + (y\cos\theta - \sin\theta)x + (y-1) = 0$
As x is real, the discriminant of this must be non-negative.
$\implies (y\cos\theta - \sin\theta)^2 - 4(y-1)^2 \geqslant 0 \implies \boxed{(y\cos\theta - \sin\theta)^2 \geqslant 4(y-1)^2}$, as required.

Consider $y^2 + 1 - (y\cos\theta - \sin\theta)^2 = y^2(1-\cos^2\theta) + 2y\cos\theta\sin\theta + 1 - \sin^2\theta$
$= (y\sin\theta)^2 + 2y\sin\theta\cos\theta + \cos^2\theta = (y\sin\theta + \cos\theta)^2 \geqslant 0$
$\implies y^2 + 1 \geqslant (y\cos\theta - \sin\theta)^2 \geqslant 4(y-1)^2 \implies \boxed{y^2 + 1 \geqslant 4(y-1)^2}$, as required.

$$\implies 3y^2 - 8y + 3 \leqslant 0 \implies \left(y - \frac{4-\sqrt{7}}{3}\right)\left(y - \frac{4+\sqrt{7}}{3}\right) \leqslant 0 \implies \boxed{\frac{4-\sqrt{7}}{3} \leqslant y \leqslant \frac{4+\sqrt{7}}{3}}.$$

(**ii**) This y value is a solution to $3y^2 - 8y + 3 = 0$ from part (i).

$$\implies y^2 + 1 = 4(y-1)^2 \implies \sqrt{y^2+1} = 2(y-1) \quad \left(\text{noting that } y - 1 = \frac{1+\sqrt{7}}{3} > 0\right).$$

Given that $y^2 + 1 \geqslant (y\cos\theta - \sin\theta)^2 \geqslant 4(y-1)^2$ and $y^2 + 1 = 4(y-1)^2$,
$y^2 + 1 = (y\cos\theta - \sin\theta)^2 \implies y^2 + 1 - (y\cos\theta - \sin\theta)^2 = (y\sin\theta + \cos\theta)^2 = 0 \implies y = -\cot\theta$

$$\implies \tan\theta = -\frac{1}{y} = \frac{-3}{4+\sqrt{7}} = \frac{\sqrt{7}-4}{3} \implies \boxed{\tan\theta = \frac{\sqrt{7}-4}{3}}.$$

And because the discriminant of the original quadratic in x is now 0,

$$x = \frac{-b}{2a} = \frac{\sin\theta - y\cos\theta}{2(y-1)} = \pm\sqrt{\frac{(\sin\theta - y\cos\theta)^2}{4(y-1)^2}} = \pm\sqrt{1} = \pm 1 \implies \boxed{x = \pm 1}.$$

5 In this question, the definition of $\binom{p}{q}$ is taken to be

$$\binom{p}{q} = \begin{cases} \dfrac{p!}{q!(p-q)!} & \text{if } p \geqslant q \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Write down the coefficient of x^n in the binomial expansion for $(1-x)^{-N}$, where N is a positive integer, and write down the expansion using the Σ summation notation.

By considering $(1-x)^{-1}(1-x)^{-N}$, where N is a positive integer, show that

$$\sum_{j=0}^{n} \binom{N+j-1}{j} = \binom{N+n}{n}.$$

(ii) Show that, for any positive integers m, n, and r with $r \leqslant m+n$,

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j}.$$

(iii) Show that, for any positive integers m and N,

$$\sum_{j=0}^{n} (-1)^j \binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n}.$$

Solution by Hauss.

(i) Coefficient of x^n in the expansion of $(1-x)^{-N}$ is $\binom{N+n-1}{n}$.

$$\implies (1-x)^{-N} = \sum_{j=0}^{\infty} \binom{N+j-1}{j} x^j.$$

Coefficient of x^{n-j} in the expansion of $(1-x)^{-1}$ is 1.
Coefficient of x^j in the expansion of $(1-x)^{-N}$ is $\binom{N+j-1}{j}$.
Coefficient of x^n in the expansion of $(1-x)^{-N-1}$ is $\binom{N+n}{n}$.

$$\implies \boxed{\sum_{j=0}^{n} (0) \binom{N+j-1}{j} = \binom{N+n}{n}}, \text{ as required.}$$

(ii) Coefficient of x^{r-j} in the expansion of $(1+x)^n$ is $\binom{n}{r-j}$.
Coefficient of x^j in the expansion of $(1+x)^m$ is $\binom{m}{j}$.
Coefficient of x^r in the expansion of $(1+x)^{m+n}$ is $\binom{m+n}{r}$.

$$\implies \boxed{\sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j} = \binom{m+n}{r}}, \text{ as required.}$$

(iii) Coefficient of x^j in the expansion of $(1+x)^{-m}$ is $(-1)^j\binom{m+j-1}{j}$.
Coefficient of x^{n-j} in the expansion of $(1+x)^{m+N}$ is $\binom{N+m}{n-j}$.
Coefficient of x^n in the expansion of $(1+x)^N$ is $\binom{N}{n}$.

$$\implies \boxed{\sum_{j=0}^{n} (-1)^j \binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n}}, \text{ as required.}$$

6 This question concerns solutions of the differential equation

$$(1-x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + k^2y^2 = k^2 \qquad (*)$$

where k is a positive integer.

For each value of k, let $y_k(x)$ be the solution of $(*)$ that satisfies $y_k(1) = 1$; you may assume that there is only one such solution for each value of k.

(i) Write down the differential equation satisfied by $y_1(x)$ and verify that $y_1(x) = x$.

(ii) Write down the differential equation satisfied by $y_2(x)$ and verify that $y_2(x) = 2x^2 - 1$.

(iii) Let $z(x) = 2(y_n(x))^2 - 1$. Show that

$$(1-x^2)\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 + 4n^2z^2 = 4n^2$$

and hence obtain an expression for $y_{2n}(x)$ in terms of $y_n(x)$.

(iv) Let $v(x) = y_n(y_m(x))$. Show that $v(x) = y_{mn}(x)$.

Solution by Mathemagicien.

(i) $\boxed{(1-x^2)(\frac{\mathrm{d}y_1}{\mathrm{d}x})^2 + y_1^2 = 1}$

If $y_1(x) = x,\ y_1(1) = 1,$ and $(1-x^2)(\frac{\mathrm{d}y_1}{\mathrm{d}x})^2 + y_1^2 = 1 - x^2 + x^2 = 1$

$\implies y_1(x) = x$ is a solution $\implies$ $\boxed{y_1(x) = x \text{ is } \textit{the} \text{ solution}}$.

(ii) $\boxed{(1-x^2)(\frac{\mathrm{d}y_2}{\mathrm{d}x})^2 + 4y_2^2 = 4}$

If $y_2(x) = 2x^2 - 1,\ y_2(1) = 1,$ and $(1-x^2)(\frac{\mathrm{d}y_2}{\mathrm{d}x})^2 + 4y_2^2 = (1-x^2)(16x^2) + 4(4x^4 - 4x^2 + 1) = 4$

$\implies y_2(x) = 2x^2 - 1$ is a solution $\implies$ $\boxed{y_2(x) = 2x^2 - 1 \text{ is } \textit{the} \text{ solution}}$.

(iii) $z(x) = 2(y_n(x))^2 - 1 \implies z^2 = 4y_n^4 - 4y_n^2 + 1, \quad \dfrac{\mathrm{d}z}{\mathrm{d}x} = 4y_n\dfrac{\mathrm{d}y_n}{\mathrm{d}x}.$

By $(*)$, $\left(\dfrac{\mathrm{d}y_n}{\mathrm{d}x}\right)^2 = \dfrac{n^2(1-y_n^2)}{1-x^2}$.

$\implies (1-x^2)(\frac{\mathrm{d}z}{\mathrm{d}x})^2 + 4n^2z^2 = 16y_n^2n^2(1-y_n^2) + 16n^2y_n^4 - 16n^2y_n^2 + 4n^2 = 4n^2$, as required.

This is the same as $(*)$ with $k = 2n$. Additionally, $z(1) = 2(y_n(1))^2 - 1 = 2 - 1 = 1$.

Hence, $z(x) = \boxed{y_{2n}(x) = 2(y_n(x))^2 - 1}$.

(iv) $v(x) = y_n(y_m(x)) \implies \dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{\mathrm{d}y_n(y_m(x))}{\mathrm{d}y_m(x)} \times \dfrac{\mathrm{d}y_m(x)}{\mathrm{d}x}$.

By $(*)$, $\left(\dfrac{\mathrm{d}y_m}{\mathrm{d}x}\right)^2 = \dfrac{m^2(1-y_m^2)}{1-x^2}$, and $\left(\dfrac{\mathrm{d}y_n(y_m(x))}{\mathrm{d}y_m(x)}\right)^2 = \dfrac{n^2(1-(y_n(y_m(x)))^2)}{1-(y_m(x))^2}$.

$$\implies \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 = \frac{n^2(1-(y_n(y_m(x)))^2)}{1-(y_m(x))^2} \times \frac{m^2(1-(y_m(x))^2)}{1-x^2} = \frac{(mn)^2(1-v^2)}{1-x^2}$$

$\implies (1-x)^2\left(\dfrac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + (mn)^2v^2 = (mn)^2$. This is the differential equation satisfied by $y_{mn}(x)$

$\implies \boxed{v(x) = y_{mn}(x)}$, as there is only one solution for each k.

7 Show that

$$\int_0^a \mathrm{f}(x)\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\mathrm{d}x, \qquad (*)$$

where f is any function for which the integrals exist.

(i) Use $(*)$ to evaluate

$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}\mathrm{d}x.$$

(ii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x}\mathrm{d}x.$$

(iii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\mathrm{d}x.$$

(iv) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{x}{\cos x(\cos x + \sin x)}\mathrm{d}x.$$

Solution by KingRS.

Substituting $u = a - x$ gives $\int_0^a \mathrm{f}(x)\mathrm{d}x = -\int_a^0 \mathrm{f}(a-u)\mathrm{d}u = \int_0^a \mathrm{f}(a-x)\mathrm{d}x$, as required.

(i) Call the integral I. The result gives $I = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{\cos x + \sin x}\mathrm{d}x \implies 2I = \frac{\pi}{2} \implies I = \boxed{\frac{\pi}{4}}$.

(ii) $\sin(\frac{\pi}{4} - x) = \frac{1}{\sqrt{2}}(\cos x - \sin x),\ \cos(\frac{\pi}{4} - x) = \frac{1}{\sqrt{2}}(\cos x + \sin x)$.

Hence, the result gives $\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x}\mathrm{d}x = \frac{1}{2}\int_0^{\frac{1}{4}\pi}(1 - \tan x)\mathrm{d}x = \frac{1}{2}\left[x + \ln\cos x\right]_0^{\frac{1}{4}\pi} = \boxed{\frac{\pi}{8} - \frac{\ln 2}{4}}$.

(iii) Using the values from (ii) with the result, and calling the integral I gives

$I = \int_0^{\frac{\pi}{4}} \ln 2\mathrm{d}x \ - \ I \implies 2I = \frac{\pi}{4}\ln 2 \implies I = \boxed{\frac{\pi}{8}\ln 2}$.

(iv) Using the result changes the integral to

$\frac{\pi}{8}\int_0^{\frac{\pi}{4}} \frac{1}{\cos x(\cos x + \sin x)}\mathrm{d}x = \frac{\pi}{8}\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{(1+\tan x)}\mathrm{d}x = \frac{\pi}{8}\left[\ln(1+\tan x)\right]_0^{\frac{1}{4}\pi} = \boxed{\frac{\pi}{8}\ln 2}$.

8 Evaluate the integral

$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}\mathrm{d}x \qquad\qquad (m > \frac{1}{2}).$$

Show by means of a sketch that

$$\sum_{r=m}^{n} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2}\mathrm{d}x, \qquad (*)$$

where m and n are positive integers with $m < n$.

(i) You are given that the infinite series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges to a value denoted by E. Use $(*)$ to obtain the following approximations for E:

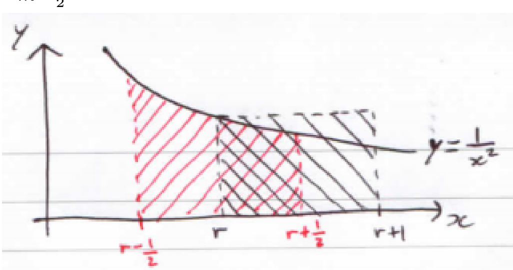
$$E \approx 2; \qquad E \approx \frac{5}{3}; \qquad E \approx \frac{33}{20}.$$

(ii) Show that, when r is large, the error in approximating $\dfrac{1}{r^2}$ by $\displaystyle\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}\mathrm{d}x$ is approximately $\dfrac{1}{4r^4}$.

Given that $E \approx 1.645$, show that $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1.08$.

Solution by Mathemagicien.

$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{2}{2m-1}. \qquad\qquad \text{For this solution, we define } \sum_{i=1}^{0} \mathrm{f}(i) = 0.$$



Clearly, $\dfrac{1}{r^2} \approx \displaystyle\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}\mathrm{d}x$ for $r \geqslant 1$. $\implies \displaystyle\sum_{r=m}^{n} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2}\mathrm{d}x$ for suitable m and n.

(i) We extend this to $\displaystyle\sum_{r=m}^{\infty} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{2}{2m-1}$, as we are given that the LHS exists.

$$\implies E = \sum_{r=1}^{\infty} \frac{1}{r^2} \approx \frac{2}{2m-1} + \sum_{r=1}^{m-1} \frac{1}{r^2} \implies E \approx 2 + 0 = \boxed{2}, \frac{2}{3} + 1 = \boxed{\frac{5}{3}}, \frac{2}{5} + 1 + \frac{1}{4} = \boxed{\frac{33}{20}}.$$

(ii) For large r, $\displaystyle\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}\mathrm{d}x - \frac{1}{r^2} = \frac{1}{r-\frac{1}{2}} - \frac{1}{r+\frac{1}{2}} - \frac{1}{r^2} = \frac{1}{4(r^4-r^2)} \approx \boxed{\frac{1}{4r^4}}$.

$$\implies \sum_{r=m}^{\infty} \left(\frac{1}{r^2} + \frac{1}{4r^4}\right) \approx \int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{2}{2m-1}.$$

$$\implies \sum_{r=1}^{\infty} \frac{1}{r^4} \approx \frac{8}{2m-1} - 4E + \sum_{r=1}^{m-1} \left(\frac{4}{r^2} + \frac{1}{r^4}\right)$$

Set $m = 3 \implies \displaystyle\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1.6 - 6.58 + 4 + 1 + 1 + 0.0625 = 1.0825 \approx \boxed{1.08}$.

9 A small bullet of mass m is fired into a block of wood of mass M which is at rest. The speed of the bullet on entering the block is u. Its trajectory within the block is a horizontal straight line and the resistance to the bullet's motion is R, which is constant.

(i) The block is fixed. The bullet travels a distance a inside the block before coming to rest. Find an expression for a in terms of m, u, and R.

(ii) Instead, the block is free to move on a smooth horizontal table. The bullet travels a distance b inside the block before coming to rest relative to the block, at which time the block has moved a distance c on the table. Find expressions for b and c in terms of M, m, and a.

Y'all have some nice energy arguments, but here's my suvat (it's basically the same).

(i) $S = a,\ U = u,\ V = 0,\ A = -\dfrac{R}{m},\ S = \dfrac{V^2 - U^2}{2A} \implies \boxed{a = \dfrac{mu^2}{2R}}$

Now, I think the question is extremely ambiguous as to whether b is relative to the block or the table, so I will denote the two possibilities b_b and b_t, respectively.

(ii) If v is the common speed of the two once the bullet comes to rest relative to the block, then by Conservation of Linear Momentum, $mu = (M+m)v \implies v = \dfrac{mu}{M+m}$.

$$S = b_t,\ U = u,\ V = \frac{mu}{M+m},\ A = -\frac{R}{m},\ S = \frac{V^2 - U^2}{2A} \implies b_t = \frac{Mm(M+2m)u^2}{2R(M+m)^2} = \frac{M(M+2m)}{(M+m)^2}a$$

$$S = c,\ U = 0,\ V = \frac{mu}{M+m},\ A = \frac{R}{M},\ S = \frac{V^2 - U^2}{2A} \implies c = \frac{Mm^2u^2}{2(M+m)^2R} = \frac{Mm}{(M+m)^2}a$$

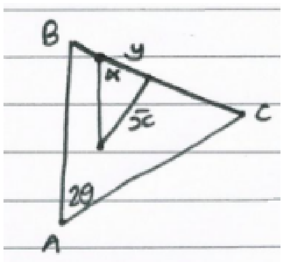
$$b_b = b_t - c = \frac{M}{M+m}a$$

$$\implies \boxed{b_t = \frac{M(M+2m)}{(M+m)^2}a},\quad \boxed{b_b = \frac{M}{M+m}a},\quad \boxed{c = \frac{Mm}{(M+m)^2}a}.$$

10 A thin uniform wire is bent into the shape of an isosceles triangle ABC, where AB and AC are of equal length at the angle at A is 2θ. The triangle ABC hangs on a small rough horizontal peg with the side BC resting on the peg. The coefficient of friction between the wire and the peg is μ. The plane containing ABC is vertical. Show that the triangle can rest in equilibrium with the peg in contact with any point on BC provided

$$\mu \geqslant 2\tan\theta(1+\sin\theta).$$

Solution by EwanClementson.



Resolving parallel and perpendicular to BC : $\quad R = mg\sin\alpha, \quad F = mg\cos\alpha$.
$\mu R \geqslant F \implies \mu mg\sin\alpha \geqslant mg\cos\alpha \implies \mu \geqslant \cot\alpha$.
α is acute and we want the smallest value of $\tan\alpha$ for our limiting value.
$\tan\alpha = \dfrac{\bar{x}}{y}$, so the limiting case is when y is as large as possible, so the peg is effectively at B.
In this case, $y = a\sin\theta$, where $a = |AB|$.
$\bar{x}$ is the perpendicular distance from BC to the centre of mass.
$\frac{1}{2}a\cos\theta(2a)\rho = (2a + 2a\sin\theta)\rho\bar{x} \implies \bar{x} = \dfrac{a\cos\theta}{2(1+\sin\theta)}$.
$\tan\alpha = \dfrac{a\cos\theta}{2(1+\sin\theta)} \div a\sin\theta = \dfrac{1}{2\tan\theta(1+\sin\theta)} \implies \cot\alpha = 2\tan\theta(1+\sin\theta)$.
$\mu \geqslant \cot\alpha \geqslant 2\tan\theta(1+\sin\theta) \implies \boxed{\mu \geqslant 2\tan\theta(1+\sin\theta)}$.

11 **(i)** Two particles move on a smooth horizontal surface. The positions, in Cartesian coordinates, of the particles at time t are $(a + ut\cos\alpha,\ ut\sin\alpha)$ and $(vt\cos\beta,\ b + vt\sin\beta)$, where a, b, u, and v are positive constants, *alpha* and β are constant acute angles, and $t \geqslant 0$.

Given that the two particles collide, show that

$$u\sin(\theta + \alpha) = v\sin(\theta + \beta),$$

where θ is the acute angle satisfying $\tan\theta = \dfrac{b}{a}$.

(ii) A gun is placed on the top of a vertical tower of height b which stands on horizontal ground. The gun fires a bullet with speed v and (acute) angle of elevation β. Simultaneously, a target is projected from a point on the ground a horizontal distance a from the foot of the tower. The target is projected with speed u and (acute) angle of elevation α, in a direction directly away from the tower.

Given that the target is hit before it reaches the ground, show that

$$2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg.$$

Explain, with reference to part (i), why the target can only be hit if $\alpha > \beta$.

Solution by Farhan.Hanif93.

(i) Equating the two pairs of components and arranging for t gives

$$t = \frac{a}{v\cos\beta - u\cos\alpha} = \frac{b}{u\sin\alpha - v\sin\beta} \implies u(a\sin\alpha + b\cos\alpha) = v(a\sin\beta + b\cos\beta)$$

$$\implies \boxed{u\sin(\theta + \alpha) = v\sin(\theta + \beta)}$$

(ii) At time t, the bullet and target have coordinates
$(vt\cos\beta,\ b + vt\sin\beta - \frac{1}{2}gt^2)$ and $(a + ut\cos\alpha,\ ut\sin\alpha - \frac{1}{2}gt^2)$, respectively.
Let T denote the time of collision, and T_0 denote the time at which the target reaches the ground.

Then $uT\sin\alpha - \frac{1}{2}gT^2 = b + vT\sin\beta - \frac{1}{2}gT^2 \implies T = \dfrac{b}{u\sin\alpha - v\sin\beta}$,
and $uT_0\sin\alpha - \frac{1}{2}gT_0^2 = 0, T_0 \neq 0 \implies T_0 = \dfrac{2u\sin\alpha}{g}$.
They collide before the target reaches the ground, so $T_0 > T \implies \boxed{2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg}$.

The coordinates of the bullet and target are equivalent to those of the two particles in part (i), as equating the y components cancels out the $-\frac{1}{2}gt^2$ term that distinguishes them.
As $2, u, \sin\alpha, b, g > 0$, $2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg \implies u\sin\alpha > v\sin\beta$.
As per (i), $u(a\sin\alpha + b\cos\alpha) = v(a\sin\beta + b\cos\beta) \implies a(u\sin\alpha - v\sin\beta) = b(v\cos\beta - u\cos\alpha)$
$\implies v\cos\beta > u\cos\alpha$ as $a, b > 0$.
$\implies \dfrac{v\cos\beta}{u\cos\alpha} > 1 > \dfrac{v\sin\beta}{u\sin\alpha} \implies \tan\alpha > \tan\beta \implies \boxed{\alpha > \beta}$, as α, β are acute.

12 Starting with the result $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$, prove that

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A) + \mathrm{P}(A \cap B \cap C).$$

Write down, without proof, the corresponding result for four events A, B, C, and D.

A pack of n cards, numbered $1,\ 2,\ \ldots,\ n$, is shuffled and laid out in a row. The result of the shuffle is that each card is equally likely to be in any position in the row. Let E_i be the event that the card bearing the number i is in the ith position in the row. Write down the following probabilities:

(i) $\mathrm{P}(E_i)$;

(ii) $\mathrm{P}(E_i \cap E_j)$, where $i \neq j$;

(iii) $\mathrm{P}(E_i \cap E_j \cap E_k)$, where $i \neq j,\ j \neq k$ and $k \neq i$.

Hence show that the probability that at least one card is in the same position as the number it bears is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}.$$

Find the probability that exactly one card is in the same position as the number it bears.

Solution by EwanClementson.

$\mathrm{P}(A \cup B \cup C) = \mathrm{P}((A \cup B) \cup C) = \mathrm{P}(A \cup B) + \mathrm{P}(C) - \mathrm{P}((A \cup B) \cap C)$.
$\mathrm{P}((A \cup B) \cap C) = \mathrm{P}((A \cap B) \cup (A \cap C)) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) - \mathrm{P}(A \cap B \cap C)$.
$\Longrightarrow \boxed{\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A) + \mathrm{P}(A \cap B \cap C)}$.

$$\boxed{\begin{gathered}\mathrm{P}(A \cup B \cup C \cup D) = \\ \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) + \mathrm{P}(B) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(A \cap D) - \mathrm{P}(B \cap C) - \mathrm{P}(B \cap D) - \mathrm{P}(C \cap D) \\ + \mathrm{P}(A \cap B \cap C) + \mathrm{P}(A \cap B \cap D) + \mathrm{P}(A \cap C \cap D) + \mathrm{P}(B \cap C \cap D) - \mathrm{P}(A \cap B \cap C \cap D)\end{gathered}}$$

(i) $\mathrm{P}(E_i) = \boxed{\dfrac{1}{n}}$ **(ii)** $\mathrm{P}(E_i \cap E_j) = \boxed{\dfrac{1}{n(n-1)}}$ **(iii)** $\mathrm{P}(E_i \cap E_j \cap E_k) = \boxed{\dfrac{1}{n(n-1)(n-2)}}$

There are $\dbinom{n}{r}$ ways for r of the n cards to be correct, with probability $\dfrac{(n-r)!}{n!}$.
P(at least one card is in the same position as the number it bears) $= \mathrm{P}(E_1 \cup E_2 \cup \cdots \cup E_n)$

$$= \binom{n}{1}\frac{1}{n} - \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{3}\frac{1}{n(n-1)(n-2)} - \cdots + (-1)^{n+1}\binom{n}{n}\frac{1}{n(n-1)(n-2)\cdots 1}$$

$$= \frac{n!(n-1)!}{1!(n-1)!n!} - \frac{n!(n-2)!}{2!(n-2)!n!} + \frac{n!(n-3)!}{3!(n-3)!n!} - \cdots + (-1)^{n+1}\frac{n!(n-n)!}{n!(n-n)!n!}$$

$= 1 - \dfrac{1}{2!} + \dfrac{1}{3!} - \cdots + (-1)^{n+1}\dfrac{1}{n!}$, as required.

P(Exactly one correct) $= n\,$P(One chosen correct, all others wrong).
P(1 chosen correct) $= \frac{1}{n}$
P($(n-1)$ all wrong) $= 1 -$ P(At least one right out of $(n-1)$) $= \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \cdots + (-1)^n\dfrac{1}{(n-1)!}$

$$\Longrightarrow \text{P(Exactly one correct)} = n \times \frac{1}{n} \times \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n\frac{1}{(n-1)!}\right)$$

$$\Longrightarrow \boxed{\text{P(Exactly one correct)} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n\frac{1}{(n-1)!}}$$

13 **(i)** The random variable X has a binomial distribution with parameters n and p, where $n = 16$ and $p = \frac{1}{2}$. Show, using an approximation in terms of the standard normal density function $\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2}$, that

$$\mathrm{P}(X = 8) \approx \frac{1}{2\sqrt{2\pi}}.$$

(ii) By considering a binomial distribution with parameters $2n$ and $\frac{1}{2}$, show that

$$(2n)! \approx \frac{2^{2n}(n!)^2}{\sqrt{n\pi}}.$$

(iii) By considering a Poisson distribution with parameter n, show that

$$n! \approx \sqrt{2\pi n}\;\mathrm{e}^{-n}\,n^n.$$

Solution by Mathemagicien.

(i) $X \sim \mathrm{B}(16, \frac{1}{2}) \approx Y \sim \mathrm{N}(8, 2^2)$.
$\implies \mathrm{P}(X = 8) \approx \mathrm{P}(7.5 < Y < 8.5) = \mathrm{P}(7.5 < 2Z + 8 < 8.5) = \mathrm{P}(-0.25 < Z < 0.25)$
$= \displaystyle\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \approx \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}}\mathrm{d}x = \frac{2}{4}\frac{1}{\sqrt{2\pi}} = \frac{1}{2\sqrt{2\pi}} \implies \boxed{\mathrm{P}(X = 8) \approx \frac{1}{2\sqrt{2\pi}}}.$

(ii) Let $A \sim \mathrm{B}(2n, \frac{1}{2}) \approx \mathrm{N}(n, \frac{n}{2}) \implies \mathrm{P}(A = n) \approx \mathrm{P}(n - \frac{1}{2} < \sqrt{\frac{n}{2}}Z + n < n + \frac{1}{2})$.
$\implies \displaystyle\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \approx \mathrm{P}\left(-\frac{1}{\sqrt{2n}} < Z < \frac{1}{\sqrt{2n}}\right) = \int_{-\frac{1}{\sqrt{2n}}}^{\frac{1}{\sqrt{2n}}} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \approx \frac{2}{\sqrt{2n}\sqrt{2\pi}}.$
$\implies \boxed{(2n)! \approx \dfrac{2^{2n}(n!)^2}{\sqrt{n\pi}}}.$

(iii) Let $E \sim \mathrm{Po}(n) \approx \mathrm{N}(n, n) \implies \mathrm{P}(E = n) = \dfrac{\mathrm{e}^{-n}\,n^n}{n!} \approx \mathrm{P}(n - \frac{1}{2} < \sqrt{n}Z + n < n + \frac{1}{2})$.
$\implies \dfrac{\mathrm{e}^{-n}\,n^n}{n!} \approx \displaystyle\int_{-\frac{1}{2\sqrt{\pi}}}^{\frac{1}{2\sqrt{\pi}}} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \approx \frac{2}{2\sqrt{n}\sqrt{2\pi}} \implies \boxed{n! \approx \sqrt{2\pi n}\;\mathrm{e}^{-n}\,n^n}.$