QUESTION 11

Part (i)

$$A=\left(a+ut\cos\alpha, ut\sin\alpha\right), B=\left(vt\cos\beta, b+vt\sin\beta\right)$$

If A and B collide at time T then:-

$$\begin{cases} a+uT\cos\alpha = vT\cos\beta \\ uT\sin\alpha = b+vT\sin\beta \end{cases} \Rightarrow \begin{cases} a=T\left(v\cos\beta-u\cos\alpha\right) \\ b=T\left(u\sin\alpha-v\sin\beta\right) \end{cases}$$

Let θ be the acute angle such that $\tan\theta=\dfrac{b}{a}$

Then $\cot\theta=\dfrac{\cos\theta}{\sin\theta}=\dfrac{v\cos\beta-u\cos\alpha}{u\sin\alpha-v\sin\beta}$

$$\Rightarrow v\sin\theta\cos\beta-u\sin\theta\cos\alpha=u\cos\theta\sin\alpha-v\cos\theta\sin\beta$$

$$\Rightarrow v\left(\sin\theta\cos\beta+\cos\theta\sin\beta\right)=u\left(\cos\theta\sin\alpha+\sin\theta\cos\alpha\right)$$

$$\Rightarrow v\sin\left(\theta+\beta\right)=u\sin\left(\theta+\alpha\right)$$

Part (ii)

Now $A=\left(a+ut\cos\alpha, ut\sin\alpha-\dfrac{1}{2}gt^2\right), B=\left(vt\cos\beta, b+vt\sin\beta-\dfrac{1}{2}gt^2\right)$

If A and B collide at time T then:-

$$uT\sin\alpha-\frac{1}{2}gT^2=b+vT\sin\beta-\frac{1}{2}gT^2 \Rightarrow b=T\left(u\sin\alpha-v\sin\beta\right)$$

Hence the time to any collision is:- $T=\dfrac{b}{u\sin\alpha-v\sin\beta}$

Using $s = ut + \frac{1}{2}gt,$ the time of flight T' of A is $0 = uT'\sin\alpha - \frac{1}{2}g\left(T'\right)^2$

$$\Rightarrow T' = \frac{2u\sin\alpha}{g}$$

For an actual collision $T' > T \Rightarrow \frac{2u\sin\alpha}{g} > \frac{b}{u\sin\alpha - v\sin\beta}$

$$\Rightarrow 2u\sin\alpha\left(u\sin\alpha - v\sin\beta\right) > bg$$

This inequality implies that $u\sin\alpha - v\sin\beta > 0 \Rightarrow \frac{u}{v} > \frac{\sin\beta}{\sin\alpha}$

Despite the differences indicated at the beginning of this part, the analysis of part (i) still applies and we have:-

$$\frac{u}{v} = \frac{\sin\left(\theta+\beta\right)}{\sin\left(\theta+\alpha\right)}$$

Hence $\frac{\sin\left(\theta+\beta\right)}{\sin\left(\theta+\alpha\right)} > \frac{\sin\beta}{\sin\alpha} \Rightarrow \sin\left(\theta+\beta\right)\sin\alpha > \sin\left(\theta+\alpha\right)\sin\beta$

$$\Rightarrow \left(\sin\theta\cos\beta + \cos\theta\sin\beta\right)\sin\alpha > \left(\sin\theta\cos\alpha + \cos\theta\sin\alpha\right)\sin\beta$$

$$\Rightarrow \sin\theta\left(\sin\alpha\cos\beta - \cos\alpha\sin\beta\right) > 0$$

$$\Rightarrow \sin\theta\sin\left(\alpha-\beta\right) > 0$$

Now θ is acute $\Rightarrow \sin\theta > 0 \Rightarrow \sin\left(\alpha-\beta\right) > 0 \Rightarrow \alpha > \beta$ (since they are both acute)

Therefore the target can only be hit if $\alpha > \beta$.