S2 UK June 2016 Model Solutions

Kprime2

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1. A student is investigating the numbers of cherries in a *Rays* fruit cake. A random sample of *Rays* fruit cakes is taken and the results are shown in the table below.

Number of cherries	0	1	2	3	4	5	⩾6
Frequency	24	37	21	12	4	2	0

(a) Calculate the mean and the variance of these data.

(3)

(b) Explain why the results in part (a) suggest that a Poisson distribution may be a suitable model for the number of cherries in a *Rays* fruit cake.

(1)

The number of cherries in a *Rays* fruit cake follows a Poisson distribution with mean 1.5

A *Rays* fruit cake is to be selected at random.

Find the probability that it contains

(c) (i) exactly 2 cherries.

(ii) at least 1 cherry.

(4)

Rays fruit cakes are sold in packets of 5

(d) Show that the probability that there are more than 10 cherries, in total, in a randomly selected packet of *Rays* fruit cakes, is 0.1378 correct to 4 decimal places.

(3)

Twelve packets of *Rays* fruit cakes are selected at random.

(e) Find the probability that exactly 3 packets contain more than 10 cherries.

(3)

1(a). Mean $= \frac{37 + 2(21) + 3(12) + 4(4) + 5(2)}{24 + 37 + 21 + 12 + 4 + 2} = 1.41$

$\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f}\right)^2$

Var $= \frac{343}{100} - (1.41)^2 = 1.4419 \approx 1.44$

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Question 1 continued $\mu \approx 1.4$ $\text{var} \approx 1.4$

(b) Mean $\approx$ Variance ≈ 1.4

$\therefore$ No. of cherries consistently occur at a rate of 1.4 per cake. $\therefore$ Poisson is suited.

(c) Let C = no. of cherries in Rays fruit cake

$C \sim Po(1.5)$

(i) $P(C=2) = \dfrac{e^{-1.5} \times 1.5^2}{2} = 0.251 \text{ (3sf)}$

(ii) $P(C \geq 1) = 1 - P(C=0)$

$= 1 - e^{-1.5} = 0.777 \text{ (3sf)}$

(d) Packs of 5 $\Rightarrow$ $1.5 \times 5 = 7.5$ cherries per pack

Let X = no. of total cherries in a selected pack of Rays fruit cake.

$X \sim Po(7.5)$

$P(X > 10) = 1 - P(X \leq 10)$.

$= 1 - 0.8622 = 0.1378 \text{ (4dp)}$

as required.

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Question 1 continued

(e) $P(X>10) = 0.1378$

Let Y = no. of packets with >10 cherries.

~~$Y \sim Bi(0.1378)$~~

$Y \sim Bi(12, 0.1378)$

$P(Y=3) = {}^{12}C_3 \times 0.1378^3 \times 0.8622^9$

$P(Y=3) = 0.152$ (3sf)

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2. In a region of the UK, 5% of people have red hair. In a random sample of size n, taken from this region, the expected number of people with red hair is 3

(a) Calculate the value of n.

(2)

A random sample of 20 people is taken from this region.
Find the probability that

(b) (i) exactly 4 of these people have red hair,

(ii) at least 4 of these people have red hair.

(5)

Patrick claims that *Reddman* people have a probability greater than 5% of having red hair. In a random sample of 50 *Reddman* people, 4 of them have red hair.

(c) Stating your hypotheses clearly, test Patrick's claim. Use a 1% level of significance.

(5)

2(a) Let X = no. of people with red hair

$X \sim B(n, 0.05)$

$E(X) = np = 0.05n = 3$

$\Rightarrow n = 60$

(b) $X \sim B(20, 0.05)$

(i) $P(X = 4) = {}^{20}C_4 \times 0.05^4 \times 0.95^{16}$

$\therefore P(X = 4) = 0.0133$ (3sf)

(ii) $P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.9841$

$P(X \geq 4) = 0.0159$

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(c) $X \sim B(50, 0.05)$

$H_0: p = 0.05$

$H_1: p > 0.05$

$P(X \geq c) < 0.01$

$\Rightarrow P(X \leq c-1) > 0.99$

$c - 1 = 7$

$\Rightarrow c = 8$

CR is $X \geq 8$

$P(X \geq 4) = 1 - 0.7604 = 0.2396$

$P(X \geq 4) = 0.2396 > 0.01$

$\therefore x = 4$ is not in critical region

$\therefore$ accept H_0

Patrick's claim is not supported by sufficient evidence. He is incorrect.

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(Total 12 marks)

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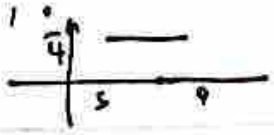
3. The random variable R has a continuous uniform distribution over the interval $[5, 9]$

(a) Specify fully the probability density function of R. **(1)**

(b) Find $P(7 < R < 10)$ **(1)**

The random variable A is the area of a circle radius R cm.

(c) Find $E(A)$ **(4)**



3(a).

$$f(r) = \begin{cases} 0 & , \text{otherwise} \\ \frac{1}{4} & , 5 \le r \le 9 \end{cases}$$

(b) $P(7 < R < 10) = \frac{2}{4} = \frac{1}{2}$

(c) $A = \pi R^2$

$E(A) = E(\pi R^2) = \pi E(R^2)$

$Var(R) = E(R^2) - [E(R)]^2 \Rightarrow$

$E(R^2) = \frac{4}{3} + 7^2$

$\therefore E(R^2) = \frac{151}{3}$

$\therefore E(A) = \frac{151}{3}\pi.$

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4. A continuous random variable X has cumulative distribution function $F(x)$ given by

$$F(x) = \begin{cases} 0 & x < 2 \\ k(ax + bx^2 - x^3) & 2 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$$

Given that the mode of X is $\frac{8}{3}$

(a) show that $b = 8$

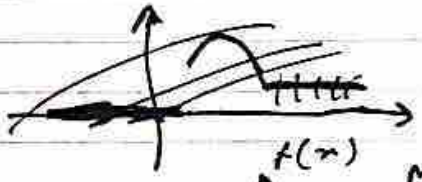
(6)

(b) find the value of k.

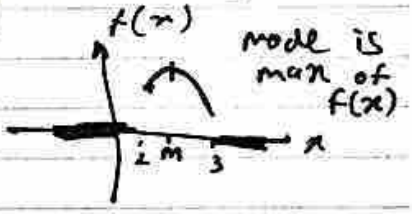
(4)

4(a). $f(x) = \frac{d}{dx}[F(x)] = k(a + 2bx - 3x^2)$

$f(x)$ is a quadratic



Mode is max value of $f(x)$



$\therefore f'(x) = k(2b - 6x)$

$f'(x) = 0 \Rightarrow k(2b - 6x) = 0$

$\therefore 2b - 6x = 0$

$\therefore x = \frac{b}{3}$

x is the mode

$\therefore$ mode $= \frac{b}{3} = \frac{8}{3}$

$\Rightarrow b = 8$ as required.

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Question 4 continued

(b) $f(3) = 1$

$\Rightarrow k(3a + 45) = 1$

$f(2) = 0 \Rightarrow k(2a + 24) = 0$

$\therefore 2a + 24 = 0$

$\Rightarrow a = -12$

$a = -12$

$\Rightarrow k = \frac{1}{3a + 45}$

$\therefore k = \frac{1}{9}$

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5. In a large school, 20% of students own a touch screen laptop. A random sample of n students is chosen from the school. Using a normal approximation, the probability that more than 55 of these n students own a touch screen laptop is 0.0401 correct to 3 significant figures.

Find the value of n.

(8)

5 Let X = no. of students with a touch screen laptop

$X \sim Bi(n, 0.2)$ $\quad np = 0.2n$

$np(1-p) = 0.2n(0.8)$

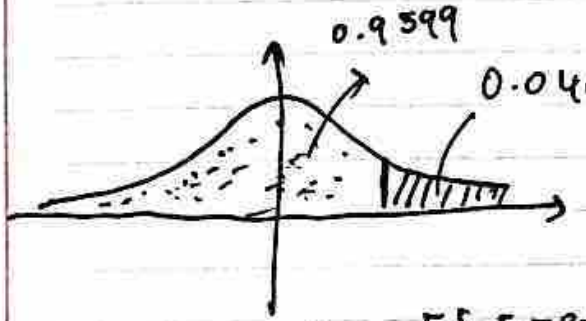
$= 0.16n$

$Y \approx$ # of students with a touch screen laptop.

$Y \sim N(0.2n, 0.16n)$

$P(X > 55) \approx P(Y \geq 55.5)$

$P(Y \geq 55.5) = P\left(Z \geq \frac{55.5 - 0.2n}{\frac{2}{5}\sqrt{n}}\right) = 0.0401$



$\Rightarrow P\left(Z \leq \frac{55.5 - 0.2n}{\frac{2}{5}\sqrt{n}}\right) = 0.9599$

$\therefore \frac{55.5 - 0.2n}{\frac{2}{5}\sqrt{n}} = 1.75$ (from tables)

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$\therefore 55.5 - 0.2n = 0.7\sqrt{n}$

$(\times 10) \Rightarrow 555 - 2n = 7\sqrt{n}$

$\therefore 2n + 7\sqrt{n} - 555 = 0$

$\therefore \sqrt{n} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$(\sqrt{n} - 15)(2\sqrt{n} + 37) = 0$

$\sqrt{n} = 15$ or $\sqrt{n} = -\frac{37}{2}$

$n = 225$ or $n = 342.25$

$\Rightarrow n = 225$

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6. A bag contains a large number of counters with one of the numbers 4, 6 or 8 written on each of them in the ratio 5 : 3 : 2 respectively.

A random sample of 2 counters is taken from the bag.

(a) List all the possible samples of size 2 that can be taken.

(2)

The random variable M represents the mean value of the 2 counters.

Given that $P(M = 4) = \frac{1}{4}$ and $P(M = 8) = \frac{1}{25}$

(b) find the sampling distribution for M.

(5)

A sample of n sets of 2 counters is taken. The random variable Y represents the number of these n sets that have a mean of 8

(c) Calculate the minimum value of n such that $P(Y \geqslant 1) > 0.9$

(3)

6(a) (4,4) (4,6) (6,4)
(6,6) (4,8) (8,4)
(8,8) (6,8) (8,6)

(b)

	4	6	8
	5/9	1/3	2/9

(b)

Counter	4	6	8
Probability	1/2	3/10	1/5

(4,4)

$P(M = 4) = \frac{1}{4}$ $P(M = 8) = \frac{1}{25}$

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Question 6 continued

$(4,4)$ | $P(M=4) = \frac{1}{4}$

~~$(6,6)$ $P(M=3) = \left(\frac{3}{10}\right)^2 = \frac{9}{100}$~~

$(6,6)$
$(4,8)$
$(8,4)$
} $P(M=6) = \left(\frac{3}{10}\right)^2 + 2\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)$
$= \frac{29}{100}$

$(4,6)$
$(6,4)$
} $P(M=5) = 2 \times \frac{1}{2} \times \frac{3}{10} = \frac{3}{10}$

$(6,8)$
$(8,6)$
} $P(M=7) = 2 \times \frac{3}{10} \times \frac{1}{5} = \frac{3}{25}$

M	4	5	6	7	8
$P(M=m)$	1/4	$\frac{3}{10}$	$\frac{29}{100}$	$\frac{3}{25}$	$\frac{1}{25}$

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(c) $P(\text{1 set of two counters has } M=8) = \frac{1}{25}$

$Y \sim Bi(n, \frac{1}{25})$

$P(Y \geq 1) = 1 - P(Y=0) = 1 - \left(\frac{24}{25}\right)^n$

$= 1$

$\therefore \quad 1 - P(Y=0) = 1 - \left(\frac{24}{25}\right)^n > 0.9$

$\therefore \quad \left(\frac{24}{25}\right)^n < 0.1$

$\therefore \quad n \ln\left(\frac{24}{25}\right) < \ln 0.1$

$\therefore \quad n > 56.405\ldots$

$\therefore n = 57$.

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7. The weight, X kg, of staples in a bin full of paper has probability density function

$$f(x) = \begin{cases} \frac{9x - 3x^2}{10} & 0 \leqslant x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Use integration to find

(a) E(X)

(4)

(b) Var (X)

(4)

(c) P($X > 1.5$)

(3)

Peter raises money by collecting paper and selling it for recycling. A bin full of paper is sold for £50 but if the weight of the staples exceeds 1.5 kg it sells for £25

(d) Find the expected amount of money Peter raises per bin full of paper.

(2)

Peter could remove all the staples before the paper is sold but the time taken to remove the staples means that Peter will have 20% fewer bins full of paper to sell.

(e) Decide whether or not Peter should remove all the staples before selling the bins full of paper. Give a reason for your answer.

(2)

7(a) $E(X) = \int_0^2 x\, f(x)\, dx$

$= \frac{1}{10} \int_0^2 9x^2 - 3x^3\, dx$

$= \frac{1}{10} \left[3x^3 - \frac{3}{4}x^4\right]_0^2$

$\therefore E(X) = \frac{6}{5}$.

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(b) $E(X^2) = \int_0^2 x^2 f(x)\,dx$

$= \frac{1}{10}\int_0^2 9x^3 - 3x^4\,dx$

$= \frac{1}{10}\left[\frac{9}{4}x^4 - \frac{3}{5}x^5\right]_0^2$

$= \frac{42}{25}$

$\therefore Var(X) = \frac{42}{25} - \left(\frac{6}{5}\right)^2 = \frac{6}{25}$

(c) $P(X > 1.5) = \int_{1.5}^2 \frac{9x - 3x^2}{10}\,dx$

$= \frac{1}{10}\left[\frac{9}{2}x^2 - x^3\right]_{1.5}^2$

$= \frac{13}{40}$

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Question 7 continued

(d) +£50 / bin full of paper

+£25 / bin if weight exceeds 1.5kg

$P(X < 1.5) = \frac{27}{40}$ $P(X > 1.5) = \frac{13}{40}$

$\therefore E(\text{money raised}) = 50 \times \frac{27}{40} + 25 \times \frac{13}{40}$

$= £41.88$.

(e) $E(\text{money raised if Peter removes all staples per bin}) = 50 \times 0.8 = £40$

$E(\text{money raised if Peter keeps the staples}) = £41.88$ from part (d)

∴ Peter should not remove all the staples before selling the bins full of paper, since profit £41.88 > £40 per bin

~~Peter~~

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