A system of n equations in n unknowns can be written in matrix form. Matrices are covered in the modules FP1 and FP2.
The matrix equation **AX = B** represents the n equations

$$a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n = b_n$$

where **A** is the matrix array of coefficients,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$

The **inverse**, $\mathbf{A}^{-1}$, of the matrix **A** is the matrix such that $\mathbf{A}\times\mathbf{A}^{-1} = \mathbf{I}$.
Then the solution of the system of equations is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.
There are various ways of finding the solution to the system of equations - finding the inverse is usually not the easiest method.

References:
NA. Ch 5
Pages 86-93

E.g. Solve the following equations by reducing the system to triangular form and back substitution.

$$\begin{aligned} x + 2y - 4z &= 3 \quad \text{(i)} \\ 2x - y + z &= 5 \quad \text{(ii)} \\ 2x + 3y + 2z &= 14 \quad \text{(iii)} \end{aligned}$$

(iii) - (ii) gives
$$\begin{aligned} x + 2y - 4z &= 3 \quad \text{(i)} \\ 2x - y + z &= 5 \quad \text{(ii)} \\ 4y + z &= 9 \quad \text{(iii)} \end{aligned}$$

(ii) - 2×(i) gives
$$\begin{aligned} x + 2y - 4z &= 3 \quad \text{(i)} \\ -5y + 9z &= -1 \quad \text{(ii)} \\ 4y + z &= 9 \quad \text{(iii)} \end{aligned}$$

(ii) + (iii) gives
$$\begin{aligned} x + 2y - 4z &= 3 \quad \text{(i)} \\ -y + 10z &= 8 \quad \text{(ii)} \\ 4y + z &= 9 \quad \text{(iii)} \end{aligned}$$

(iii) + 4(ii) gives $41z = 41$
i.e. $z = 1$.
Substitute in (ii) gives $y = 2$ and then into (i) gives $x = 3$

E.g. the above process represented in augmented matrix notation gives the following.

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 2 & -1 & 1 & 5 \\ 2 & 3 & 2 & 14 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 2 & -1 & 1 & 5 \\ 0 & 4 & 1 & 9 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 0 & -5 & 9 & -1 \\ 0 & 4 & 1 & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 0 & -1 & 10 & 8 \\ 0 & 4 & 1 & 9 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 0 & -1 & 10 & 8 \\ 0 & 0 & 41 & 41 \end{array}\right)$$

For work on matrices, see Topic 1 of FP1.

References:
NA. Ch 5
Pages 93-100

Gaussian elimination
If any one equation is added to another then the resulting system is slightly different, but has the same solution.
By performing a number of such operations, the matrix of coefficients may be *reduced* to triangular form. This will give a value to one variable which may be substituted back to give all the others.

References:
NA. Ch 5
Pages 101-113

Example 5.5
Page 107

Exercise 5
Q. 8

E.g. Solve the equations
$$\begin{aligned} x + 2y + z &= 1 \\ 3x - y + 2z &= 3.5 \\ 2x + 3y - z &= -1.5 \end{aligned}$$
Using the notation above, but first rearranging the equations so that the second is written first gives the following spread-sheet.

	3	-1	2	3.5	
	2	3	-1	-1.5	
	1	2	1	1	0.25
0.6666667	0	3.6666667	-2.333333	-3.8333333	
0.3333333	0	2.3333333	0.3333333	-0.1666667	-0.25
0.6363636		0	1.8181818	2.2727273	1.25

i.e. $x = 0.25$, $y = -0.25$, $z = 1.25$

Notation
In carrying out the process on paper and on a spreadsheet it is convenient to simply write out the coefficients of the variables and the column matrix, **b**.
This is usually done in what is known as the *Augmented matrix* $(\mathbf{A} \mid \mathbf{b})$.

Stability
When working on a spreadsheet to produce a solution involving decimals, the principle should be to add or subtract multiples of rows where the multiple is less than 1.
This can be done by identifying the row with the largest first coefficient. This is called the *pivot*.
Without pivoting the method is unstable.

References:
NA. Ch 5
Pages

Exercise 5
Q. 9 (i), (iv)

Numerical Computation
Version A: page 9
Competence statements m1

Systems of linear equations

Two equations in two unknowns can be solved by simultaneous equations methods. It is possible, but very tedious, to solve three equations in three unknowns by the same process.
In FP1 you will have seen solutions using matrix algebra.

References: NA. Ch 6 Pages 125-127

E.g. $\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & -1 & 4 \end{pmatrix}$; $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$

where $\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$; $\mathbf{U} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

Definitions

You should refer to the work in FP1 for the definitions and properties of matrices and methods of finding their inverse.
The following definitions apply only to square matrices.
A **diagonal matrix, D,** is one where every element other than those in the leading diagonal are zero.
A **lower triangular matrix, L,** is one where all elements in the leading diagonal and above are zero.
An **Upper triangular matrix, U,** is one where all elements in the leading diagonal and below are zero.
Every matrix, **A**, can be written
A = **D** + **L** + **U**.

The Gauss-Jacobi method

For the system of equations **Ax = b**:

- Rewrite **A** = **D** + **L** + **U**
- Rewrite the system **Dx = b – (L + U)x**
- Decide on a starting value $\mathbf{x}^0$.
- Solve using $\mathbf{D}\mathbf{x}^{r+1} = \mathbf{b} - (\mathbf{L}+\mathbf{U})\mathbf{x}^r$.

If, in each row of the matrix of coefficients, A, the magnitude of the entry in the leading diagonal is greater than or equal to the sum of the magnitudes of all other elements in the row, then the matrix is said to be **diagonally dominant.** It can be shown that if **A** is diagonally dominant with at least one inequality being strict, then the Jacobi method converges. (The method may also converge even if the matrix is not diagonally dominant.)

References: NA. Ch 6 Pages 127-128

Example 6.2 Page 127

E.g. Solve the following equations.

$3x + y - 2z = 3$

$x + 5y + z = 5$

$2x - 4y + 7z = 8$

The augmented matrix is $\left(\begin{array}{ccc|c} 3 & 1 & -2 & 3 \\ 1 & 5 & 1 & 5 \\ 2 & -4 & 7 & 8 \end{array}\right)$

Rewrite the equations

$3x = 3 - y + 2z \Rightarrow x = \dfrac{3 - y + 2z}{3}$

$5y = 5 - x - z \Rightarrow y = \dfrac{5 - x - z}{5}$

$7z = 8 - 2x + 4y \Rightarrow z = \dfrac{8 - 2x + 4y}{7}$

Rewriting the equations in the form $3x =$ ….., $5y =$ …… and $7z =$ …. is the form **Dx = …..** .

Starting with (0,0,0) gives $(x_1, y_1, z_1) = \left(1, 1, \frac{8}{7}\right)$

$\Rightarrow x_2 = \dfrac{3 - 1 + \frac{16}{7}}{3} = \dfrac{30}{21} = 1.4286,\ y_2 = 0.5714,\ z_2 = 1.4286$

$\Rightarrow x_3 = 1.7619, y_3 = 0.4286, z_3 = 1.0612$

This converges, after 28 iterations to:

$x = 1.5, y = 0.5, z = 1$

The Gauss-Seidel method

In the Jacobi method all three values are found using the values from the previous Iteration.
The process will usually converge faster if the latest value of each variable is used at each step. That is, the improved value of x is used in finding y and the improved values of x and y are used to find z.

References: NA. Ch 6 Pages 129-130

Example 6.3 Page 129

Exercise 6 Q. 8, 9(i), 10

E.g. Solve the above equations using the Gauss-Seidel method.

0	0	0
1	0.8	1.314286
1.609524	0.415238	0.920272
1.475102	0.520925	1.019071
1.505739	0.495038	0.995525
1.498671	0.501161	1.001043
1.500309	0.49973	0.999757
1.499928	0.500063	1.000056
1.500017	0.499985	0.999987
1.499996	0.500003	1.000003
1.500001	0.499999	0.999999
1.5	0.5	1

The spreadsheet shows that the solution is found in 12 iterations instead of 28.