(4) $(1+x)^n(1x+x)^n \equiv (1+x)^{2n} \Rightarrow \left(\sum_{=0}^{n}\binom{n}{s}x^s\right)^2 = \sum_{t=0}^{2n}\binom{2n}{t}x^t$

Comparing coefficients of x^n on left side we have $\sum_{s=0}^{n}\binom{n}{s}\binom{n}{n-s}$ and on right $\binom{2n}{n}$

But $\binom{n}{\ }=\binom{n}{b-s}$ by symmetry so $\sum_{s=0}^{n}\binom{n}{s}^2=\binom{2n}{n}$

(i) Using the same method with $(1+x)^n(1-x)^n=(1-x^2)^n$

we have $\left(\sum_{=0}^{n}\binom{n}{s}x^s\right)\left(\sum_{=0}^{n}\binom{n}{s}(-x)^s\right)^2 = \sum_{t=0}^{n}\binom{n}{t/2})(-x^2)^{t/2}$

$$\Rightarrow \sum_{s=0}^{n}\binom{n}{s}\binom{n}{n-s}(-1)^{n-s}=\binom{n}{n/2}(-1)^{n/2}$$

but $\binom{n}{s}=\binom{n}{n-s}$ hence $\sum_{r=0}^{n}\binom{n}{s}^2(-1)^r=\binom{n}{n/2}(-1)^{n/2}$ by writing $r=n-s$

(ii) Considering again $(1+x)^n(1+x)^{n-1}\equiv(1+x)^{2n-1}$

$\Rightarrow \left(\sum_{t=0}^{n}\binom{n}{t}x^t\right)\sum_{s=0}^{n-1}\binom{n-1}{s}x^s=\sum_{r=0}^{2n-1}\binom{2n-1}{r}x^r$, so equating coefficients of x^n

$$\sum_{t=1}^{n}\binom{n}{t}\binom{n-1}{n-t}=\binom{2n-1}{n}\Rightarrow\sum_{t=1}^{n}\binom{n}{t}\tfrac{t}{n}\binom{n}{t}=\tfrac{n}{2n}\binom{2n}{n}\Rightarrow\sum_{t=1}^{n}2t\binom{n}{s}^2=n\binom{2n}{n}$$

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(8) $2\sin\frac{1}{2}\theta\sum_{k=0}^{n}\sin k\theta=\sum_{k=0}^{n}2\sin k\theta\sin\frac{1}{2}\theta=\sum_{k=0}^{n}\left[\cos\left(k-\frac{1}{2}\right)\theta-\cos\left(k+\frac{1}{2}\right)\theta\right]$

$=\cos\left(-\frac{1}{2}\theta\right)-\cos\left(n+\frac{1}{2}\right)\theta$ all other terms cancelling

hence, $\sum_{k=0}^{n}\sin k\theta=\dfrac{\cos\frac{1}{2}\theta-\cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$

(i) $\sum_{k=0}^{n}\sin\frac{k\pi}{n}=\dfrac{\cos\frac{\pi}{2n}-\cos\left(n+\frac{1}{2}\right)\frac{\pi}{n}}{2\sin\frac{\pi}{2n}}=\dfrac{-2\sin\left(\frac{n+1}{2n}\right)\pi\sin\left(-\frac{\pi}{2n}\right)}{2\sin\frac{\pi}{2n}}=\dfrac{\sin\left(\frac{n+1}{2n}\right)\pi\cos\frac{\pi}{2}}{\sin\frac{\pi}{2n}}$

for large n, $\left(\frac{n+1}{2n}\right)\pi=\frac{\pi}{2}+\frac{\pi}{2n}$, $\cos\frac{\pi}{2}=1$ and $\sin\frac{\pi}{2n}\approx\frac{\pi}{2n}$, $\cos\frac{\pi}{2n}\approx 1$

hence, $\sum_{k=0}^{n}\sin\frac{k\pi}{n}=\dfrac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\approx\dfrac{1}{\frac{\pi}{2n}}=\dfrac{2n}{\pi}$

(ii) Differentiating the first result with respect to θ we have

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$$\sum_{k=0}^{n} k\cos k\theta = \frac{2\sin\left(\frac{\theta}{2}\right)\left[-\frac{1}{2}\sin\left(\frac{\theta}{2}\right)+\left(n+\frac{1}{2}\right)\sin\left(n+\frac{1}{2}\right)\theta\right]-\cos\frac{\theta}{2}\left[\cos\frac{1}{2}\theta-\cos\left(n+\frac{1}{2}\right)\theta\right]}{4\sin^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{-\sin^2\frac{\theta}{2}+(2n+1)\sin\frac{\theta}{2}\sin\left(n+\frac{1}{2}\right)\theta-\cos^2\frac{\theta}{2}+\cos\frac{\theta}{2}\cos\left(n+\frac{1}{2}\right)\theta}{4\sin^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{-1+2n\sin\frac{\theta}{2}\sin\left(n+\frac{1}{2}\right)\theta+\cos n\theta}{4\sin^2\left(\frac{\theta}{2}\right)}$$

so $\displaystyle\sum_{k=0}^{n} k\left[1-2\sin^2\left(\frac{k\theta}{2}\right)\right] = \frac{-1+2n\sin\frac{\theta}{2}\sin\left(n+\frac{1}{2}\right)\theta+\cos n\theta}{4\sin^2\left(\frac{\theta}{2}\right)}$

$$\Rightarrow \sum_{k=0}^{n} k\sin^2\left(\frac{k\theta}{2}\right) = \frac{1-2n\sin\frac{\theta}{2}\sin\left(n+\frac{1}{2}\right)\theta-\cos n\theta}{8\sin^2\left(\frac{\theta}{2}\right)} + \frac{1}{2}\sum_{k=0}^{n} k$$

$$= \frac{1-2n\sin\frac{\theta}{2}\sin\left(n+\frac{1}{2}\right)\theta-\cos n\theta}{8\sin^2\left(\frac{\theta}{2}\right)} + \frac{1}{4}n(n+1)$$

so $\displaystyle\sum_{k=0}^{n} k\sin^2\left(\frac{k\pi}{2n}\right) = \frac{1-2n\sin\left(\frac{\pi}{2n}\right)\sin\left(n+\frac{1}{2}\right)\frac{\pi}{n}-\cos\pi}{8\sin^2\left(\frac{\pi}{2n}\right)} + \frac{1}{4}n(n+1)$

$$\approx \frac{2+2n\times\sin^2\frac{\pi}{2n}}{8\times\sin^2\frac{\pi}{2n}} + \frac{1}{4}n(n+1) = \frac{(\pi-2)n^2}{2\pi^2} + \frac{1}{4}n(n+1)$$

$$= \frac{2}{8}\left(\frac{2n}{\pi}\right)^2 + \frac{n}{4} + \frac{n^2}{4} + \frac{n}{4} \approx \frac{n^2}{\pi^2} + \frac{n^2}{4} = \left(\frac{1}{4}+\frac{1}{\pi^2}\right)n^2 \text{ for large } n$$

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(9) (i) $\frac{dv}{dt} = -k_1x^3$ v positive when directed away from G so $v\frac{dv}{dx} = -k_1x^3$

hence $\frac{1}{2}v^2 = -\frac{1}{4}k_1x^4 + \frac{1}{2}u^2$ by initial condition $v = u$ when $x = 0$

i.e. $v^2 = u^2 - \frac{1}{2}k_1x^4$ for $x < R$

(ii) clearly, $u^2 \geq \frac{1}{2}k_1x^4$ so maximum distance X from G is given by $X^4 = \frac{2u^2}{k_1}$

So fragment does not reach R if $R^4 > \frac{2u^2}{k_1} \Rightarrow \frac{2a^2}{k_1} > \frac{2u^2}{k_1}$ or $a > u$ since a & u are both > 0

(iii) Now suppose that $u > a$ so fragment reaches $x = R$ with velocity u_1 say, so $u_1^2 = u^2 - \frac{1}{2}k_1R^4$

subsequently, $v\frac{dv}{dx} = -\frac{k_2}{x^4} \Rightarrow \frac{1}{2}v^2 = \frac{k_2}{3x^3} - \frac{k_2}{3R^3} + \frac{1}{2}u_1^2$

i.e. $v^2 = u_1^2 + 2\left(\frac{k_2}{3x^3} - \frac{k_2}{3R^3}\right) = u^2 - \frac{1}{2}k_1R^4 + 2\left(\frac{k_2}{3x^3} - \frac{k_2}{3R^3}\right)$

Fragment is always moving away from G if $v > 0$ for all x i.e. since $\frac{k_2}{3x^3} > 0$

for $u^2 > \frac{1}{2}k_1R^4 + \frac{2k_2}{3R^3} \Rightarrow 6u^2 \geq 3k_1R^4 + \frac{4k_2}{R^3} = 6b^2 \Rightarrow u \geq b$

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(iv) Maximum distance from G will be when $v = 0$, let this distance be X then

$$u^2 - \tfrac{1}{2}k_1R^4 + 2\left(\frac{k_2}{3X^3} - \frac{k_2}{3R^3}\right) = 0 \Rightarrow \frac{2k_2}{3X^3} = \tfrac{1}{2}k_1R^4 - u^2 + \frac{2k_2}{3R^3} = b^2 - u^2$$

$$\Rightarrow X^3 = \frac{2k_2}{3(b^2-u^2)} \text{ so maximum distance is } \sqrt[3]{\frac{2k_2}{3(b^2-u^2)}}$$

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