MEI

Mathematics in Education and Industry

MEI STRUCTURED MATHEMATICS

METHODS FOR ADVANCED MATHEMATICS, C3

Practice Paper C3-D

Additional materials: Answer booklet/paper
Graph paper
List of formulae (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS

- Write your Name on each sheet of paper used or the front of the booklet used.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION

- The number of marks is given in brackets [] at the end of each question or part-question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is **72.**
- **You are reminded of the need for clear presentation in your answers.**

Section A (36 marks)

1 You are given that $y^2 = 4x + 7$.

(i) Use implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y. [2]

(ii) Make x the subject of the equation.

Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ and hence show that in this case $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$. [3]

2 **(i)** Expand $\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)^2$. [1]

(ii) Hence find $\int \left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)^2 \mathrm{d}x$. [3]

3 **(i)** Sketch the graph of $y = |3x - 6|$. [2]

(ii) Solve the equation $|3x - 6| = x + 4$ and illustrate your answer on your graph. [4]

4 Find $\int x \sin 3x \,\mathrm{d}x$. [4]

5 Make x the subject of $t = \ln \sqrt{\frac{5}{(x-3)}}$. [4]

6 The function f(x) is defined as $\mathrm{f}(x) = \frac{\ln x}{x}$. The graph of the function is shown in Fig. 6.

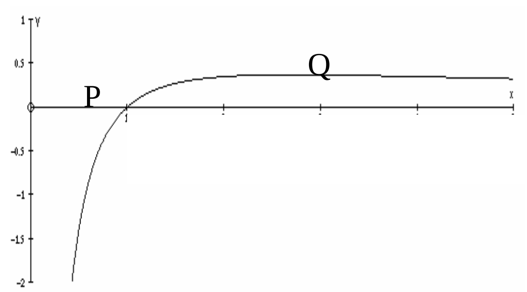

Fig. 6

(i) Give the coordinates of the point, P, where the curve crosses the x-axis. [1]

(ii) Use calculus to find the coordinates of the stationary point, Q, and show that it is a maximum. [6]

7 An oil slick is circular with radius r km and area A km^2. The radius increases with time at a rate given by $\dfrac{\mathrm{d}r}{\mathrm{d}t} = 0.5$, in kilometres per hour.

(i) Show that $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \pi r$. [4]

(ii) Find the rate of increase of the area of the slick at a time when the radius is 6 km. [2]

Section B (36 marks)

8 Fig. 8 shows the graph of $y = x\sqrt{1+x}$. The point P on the curve is on the x-axis.

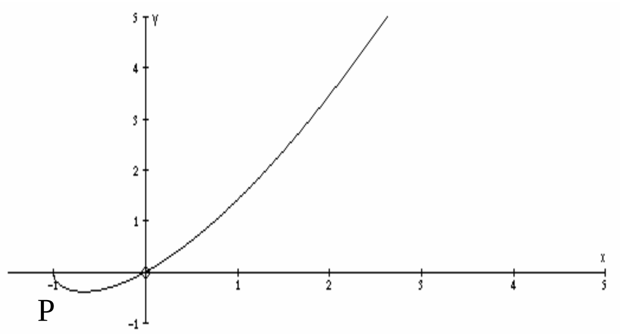

Fig. 8

(i) Write down the coordinates of P. [1]

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x+2}{2\sqrt{1+x}}$. [4]

(iii) Hence find the coordinates of the turning point on the curve.
What can you say about the gradient of the curve at P? [4]

(iv) By using a suitable substitution, show that $\displaystyle\int_{-1}^{0} x\sqrt{1+x}\,\mathrm{d}x = \int_{0}^{1} \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) \mathrm{d}u$.

Evaluate this integral, giving your answer in an exact form.

What does this value represent? [7]

(v) Use your answer to part (ii) to differentiate $y = x\sqrt{1+x}\sin 2x$ with respect to x.
(You need not simplify your result.) [2]

9 The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = x^2, \qquad g(x) = 2x - 1,$$

for all real values of x.

(i) State the ranges of $f(x)$ and $g(x)$.
Explain why $f(x)$ has no inverse. [3]

(ii) Find an expression for the inverse function $g^{-1}(x)$ in terms of x.
Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the same axes. [4]

(iii) Find expressions for $gf(x)$ and $fg(x)$. [2]

(iv) Solve the equation $gf(x) = fg(x)$.

Sketch the graphs of $y = gf(x)$ and $y = fg(x)$ on the same axes to illustrate your answer. [4]

(v) Show that the equation $f(x + a) = g^2(x)$ has no solution if $a > \frac{1}{4}$. [5]

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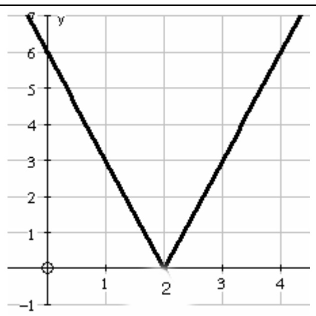
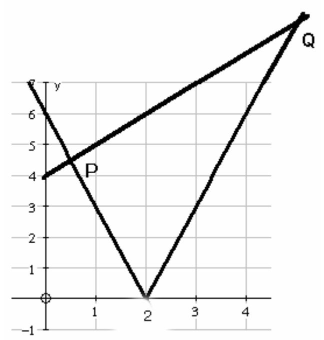
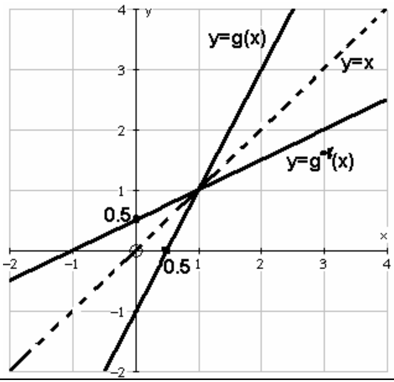
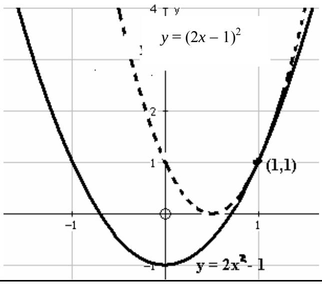
MARK SCHEME

Qu		Answer	Mark	Comment
Section A				
1	**(i)**	$y^2 = 4x + 7 \Rightarrow 2y.\dfrac{dy}{dx} = 4$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{2}{y}$	M1 A1 **2**	
	(ii)	$x = \frac{1}{4}\left(y^2 - 7\right) \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{4}.2y = \dfrac{y}{2} = \dfrac{1}{\frac{2}{y}}$	B1 M1 A1 **3**	
2	**(i)**	$e^{2x} + 2 + e^{-2x}$	B1 **1**	
	(ii)	$= \int (e^{2x} + 2 + e^{-2x})dx$ $= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$	B1 B1 B1 **3**	One for each exponential term, one for both 2x and constant.
3	**(i)**		B1 B1 **2**	One for two half lines; one for correct orientation and meeting at $(2,0)$.
	(ii)	 At P $-3x + 6 = x + 4$ $\Rightarrow \quad x = \frac{1}{2}$ At Q $3x - 6 = x + 4$ $\Rightarrow \quad x = 5$ The solution is $x = \frac{1}{2}, 5$. As shown on graph	M1 A1 A1 E1 **4**	
4		$\int x \sin 3x \, dx; \quad u = x \Rightarrow \dfrac{du}{dx} = 1, \ \dfrac{dv}{dx} = \sin 3x \Rightarrow v = -\dfrac{1}{3}\cos x$ $= -\frac{1}{3}x\cos 3x + \int \frac{1}{3}\cos 3x \, dx$ $= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + c$	M1 A1 M1 A1 **4**	Choice of u $-\frac{1}{3}\cos 3x$ Correct form c must be seen

5		$t=\ln\sqrt{\frac{5}{(x-3)}}$ $=-\frac{1}{2}\ln\frac{(x-3)}{5}$ $\Rightarrow -2t=\ln\frac{(x-3)}{5}$ $\Rightarrow e^{-2t}=\frac{(x-3)}{5}\Rightarrow x=5e^{-2t}+3$	M1 M1 A1 A1 **4**	Rules of logs Change to exponentials
6	**(i)**	P(1,0)	B1 **1**	
	(ii)	$\frac{dy}{dx}=\frac{x.\frac{1}{x}-1.\ln x}{x^2}$ $=\frac{1-\ln x}{x^2}$ At Q, gradient is zero, so $x=e$. Q is $(e,\frac{1}{e})$. $\frac{d^2y}{dx^2}=\frac{x^2(-\frac{1}{x})-(1-\ln x).2x}{x^4}$ $=\frac{-3+2\ln x}{x^3}$ When $x=e$, this is –ve, so Q is a maximum.	M1 A1 M1 A1 M1 A1 **6**	quotient rule $=0$ Or equivalent methods For $^1/_e$.
7	**(i)**	$A=\pi r^2\Rightarrow\frac{dA}{dr}=2\pi r$ $\Rightarrow\frac{dA}{dt}=\frac{dA}{dr}.\frac{dr}{dt}=\frac{1}{2}\times 2\pi r=\pi r$	M1 A1 M1 A1 **4**	
	(ii)	So when r = 6 , $\frac{dA}{dt}=6\pi\ (=15.707...)$. The area increases at 15.7 km^2h^{-1}, to 3sf.	M1 A1 **2**	Substituting

Section B				
8	(i)	$\mathrm{P}(-1,0)$	B1 **1**	
	(ii)	$y = x\sqrt{1+x}$ $= x(1+x)^{\frac{1}{2}}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1.(1+x)^{\frac{1}{2}} + x.\frac{1}{2}(1+x)^{-\frac{1}{2}}$ $= \frac{2(1+x)+x}{2(1+x)^{\frac{1}{2}}}$ $= \frac{3x+2}{2\sqrt{1+x}}$	M1 A1 M1 E1 **4**	Product rule Any correct expression Combining fractions
	(iii)	At a turning point, gradient is zero. $x = -\frac{2}{3}$ there. Then $y = -\frac{2}{3}\sqrt{\frac{1}{3}}$ $= -\frac{2\sqrt{3}}{9}$ These are the coordinates of the TP. At P the gradient is undefined.	M1 A1 A1 B1 **4**	Accept a simplified form with √3 in the bottom. Accept reference to infinity or to vertical.
	(iv)	$\int_{-1}^{0} x\sqrt{1+x}dx$ $= \int_{0}^{1}(u-1)u^{\frac{1}{2}}du, \quad u = 1+x$ $\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ $= \int_{0}^{1}(u^{\frac{3}{2}} - u^{\frac{1}{2}})du$ $= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{0}^{1}$ $= -\frac{4}{15}$ The magnitude represents the area bounded by the curve and axis between P and O. It is negative because the curve is below the axis (except at the end points).	M1 E1 M1A1 A1 B1 B1 **7**	Integral in *u* Change of limits Integrating One for geometry and one for sign.
	(v)	$y = x\sqrt{1+x}\sin 2x$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x+2}{2\sqrt{1+x}}\sin 2x + x\sqrt{1+x}.2\cos 2x$	M1A1 **2**	

9	**(i)**	Range of f is $[0,\infty)$, of g $(-\infty,\infty)$. f has no inverse because (say) for any value of f > 0 there are 2 corresponding values of x	B1 B1 E1 **3**	
	(ii)	$y = 2x - 1$ $\Rightarrow x = \frac{1}{2}y + \frac{1}{2}$ $\Rightarrow g^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$ 	M1 A1 B1 B1 **4**	One mark for one line, and one mark for second correctly related
	(iii)	$gf(x) = 2x^2 - 1$ $fg(x) = (2x-1)^2$	B1 B1 **2**	
	(iv)	$2x^2 - 1 = (2x-1)^2$ $\Rightarrow 0 = 2x^2 - 4x + 2$ $\Rightarrow 0 = 2(x-1)^2$ $\Rightarrow x = 1$ 	M1 A1 B1 B1 **4**	$y = (2x-1)^2$ $y = (2x-1)^2$
	(v)	$f(x+a) = (x+a)^2$ $g^2(x) = 2(2x-1) - 1 = 4x - 3$ $f(x+a) = g^2(x) \Rightarrow (x+a)^2 = 4x - 3$ $\Rightarrow x^2 + (2a-4)x + a^2 + 3 = 0$ $\Rightarrow$ There are two roots to this equation if $(2a-4)^2 > 4(a^2+3)$ i.e. $4a^2 - 16a + 16 > 4a^2 + 12$ $\Rightarrow 16a < 4 \Rightarrow a < \frac{1}{4}$	B1 M1 M1 A1 A1 **5**	Both fns correct Equating Using $b^2 - 4ac$ Correct inequality Result