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Introduction

If you enjoy Mathematics and are intending to continue your study at University level, then it is likely that you are enthusiastic enough to do a bit of independent learning before you show up for your first lecture. It is not a myth that the leap from A2 Level (or equivalent) to University level Mathematics can be extremely challenging: whilst the former involves a large amount of computation based on assumed results, the latter tends to be more concerned with the rigourous derivation and proof of such results. This change of focus, along with the introduction of unfamiliar branches of Maths, can seem daunting. So we figured we should make a booklet to provide a brief introduction to some of the topics you will inevitably meet in a Mathematics degree. We've tried (really hard) to do this without being:

1. boring,
2. complicated, or
3. scary (heaven forbid).

If you are using this booklet in preparation for a University interview (which may not be a bad idea), then we recommend the first five sections in particular as, not only do those topics lay the foundations for a study of Maths, it will also be in massively your favour to demonstrate a basic understanding of proofs.

At several points throughout the text, we've suggested that you do some proofs for yourself. Do them! You'll find it really helpful to get an early grip on proofs, and there's no better way to learn maths than through doing it (we have tried and tested this hypothesis and it's definitely true!). In most cases, we've included said proofs anyway, so you can check your answers or have a look if you get stuck. But don't be tempted just to read through them before you try them...

Also, make sure you find *The Maths Square* on Facebook!

The URL is *www.facebook.com/TheMathsSquare*.

We've set up a page where links will be provided to free PDF downloads of the material in this book, and we'll keep updating what's available, adding introductions to other topics you'll be meeting at University. The group will also provide you with an opportunity to discuss maths directly with other students in the same boat as you, as well as advice and help from a couple of delightful final year students (us).

1 A Bit of Notation...

One large problem when you come to study Maths at University is the sudden influx of weird and wonderful symbols. I say 'weird' because, frankly, some of them are a bit funky (I draw your attention to $\aleph$), and 'wonderful' because, as you'll soon come to realise, they have the potential to save you an awful lot of time when answering problems or formulating proofs. So, without further ado, here is reasonably comprehensive list of some common mathematical symbols.

1.1 Greek Letters

Greek letters crop up all over the place in Maths. They're frequently used to represent constants (things that stay the same), variables (things that change), and even the odd function here and there. It is worth noticing that upper and lower case letters are **not** interchangeable; for example, δ can definitely not be haphazardly replaced by Δ.

A small disclaimer: It is not worth memorising the Greek letters! At least some of them will, hopefully, be familiar to you already: θ and ϕ are common names for our favourite angles, and who could forget our good friend π?

Name	Lower Case	Upper Case
alpha	α	A
beta	β	B
gamma	γ	Γ
delta	δ	Δ
epsilon	ε	E
zeta	ζ	Z
eta	η	H
theta	θ	Θ
iota	ι	I
kappa	κ	K
lambda	λ	Λ
mu	μ	M
nu	ν	M
xi	ξ	Ξ
o	o	O
pi	π	Π
rho	ρ	P
sigma	σ	Σ
tau	τ	T
upsilon	υ	Υ
phi	ϕ	Φ
chi	χ	X
psi	ψ	Ψ
omega	ω	Ω

1.2 Some Common Mathematical Symbols

You don't need to learn these now! They are just here for quick reference, and all of them are introduced in a lot more detail when we use them.

Symbol	What it means...
$p \vee q$	p or q
$p \wedge q$	p and q
$\neg p$	not p
$p \Rightarrow q$	p implies q
$p \Leftrightarrow q$	p if and only if q
$p \equiv q$	p is equivalent to q
$\forall$	for all
$\exists$	there exists
$\in$	is an element of
$\mid$	such that
$\lvert A \rvert$, A a set	cardinality of A
$A \subseteq B$	A a subset of B
$A \subset B$	A a strict subset of B
$A \cup B$	the union of A and B
$A \cap B$	the intersection of A and B
A^c	the complement of A
$A \setminus B$	the set difference of A and B

2 Logic

A lot of what we do at University level Mathematics involves proving things; in order to know when we've proved something, we need to have a good understanding of logic. So here are the basics!

2.1 Definitions

Before we start, we need to get to grips with some definitions and notation.

- A *predicate* is a property that a variable may or may not have.
- A *proposition* is a statement that is either true (T), or false (F).

For example, if $\phi(n)$ means 'n is an even number', then ϕ is the predicate and $\phi(n)$ for given is the proposition (as it is either T or F). So $\phi(4)$, '4 is an even number', is T. You may now be able to see that $\phi(3)$ is F...

Propositions are also often called *sentences* or *statements*. For example, $\phi(7)$ is a false statement.

Note that predicates and propositions can involve more than one variable. So, letting $\psi(m, n)$ be the proposition that m is a multiple of n, we have $\psi(6, 2)$ is T and $\psi(2, 6)$ is F.

2.2 Logical Connectives

Logical connectives provide ways to connect predicates. This allows us to make compound statements which can be T or F depending on whether the individual statements being connected are T or F.

- $p \vee q$ means "p or q". This is true if at least one of p and q is true, and is false otherwise. This is called the *disjunction* or p and q.
- $p \wedge q$ means "p and q". This is true if both p and q is true, and is false otherwise. This is called the *conjunction* or p and q.
- $\neg p$ means "not p". This is T if p is F, and F is p is T. This is called the *negation* of p.
- $p \Rightarrow q$ means "p implies q". This is:

  true if p is T and q is T,
false if p is T and q is F,
true if p is F (regardless of whether q is T or F).

- $p \Leftrightarrow q$, means "p if and only if q". This is:

  true if p, q both T,
true if p, q both F,
F otherwise.

For example, define:

$$\phi(n) : \text{“}n \text{ is an even number”}$$
$$\psi(n) : \text{“}m \text{ is a multiple of } n\text{”}$$

Then,

$$\phi(4) \wedge \psi(2,6) \text{is } T,$$
$$\phi(4) \vee \psi(2,6) \text{is } F.$$

Also,

- $\phi(n) \Rightarrow \phi(n+2)$ is T (“n is even $\Rightarrow$ $(n+2)$ is even”);
- $\phi(n) \Rightarrow \phi(n+1)$ is F (“n is even $\Rightarrow$ $(n+1)$ is even”, but this is obviously false!);
- $\phi(2n) \Rightarrow \phi(n)$ is F ($n = 3$ is a counter example of this);
- $\phi(2n-1) \Rightarrow \phi(n)$ is T (since $\phi(2n-1)$ is F, so $\phi(n)$ is irrelevant).

2.3 Truth Tables

A *truth table* is a table which lists all of the possible combinations of T and F. They provide a great way to illustrate the definitions above:

p	q	$p \wedge q$	$p \vee q$	$\neg p$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	F	T	T
T	F	T	F	F	F	F
F	T	T	F	T	T	F
F	F	F	F	T	T	T

The values in the first two columns are called the *truth inputs*, and they determine the values in the following columns. So, if p was T and q was F then, looking at the second row, we have that $p \vee q$ is T.

Truth tables make complicated statements easier to decipher.

Example

When is $p \vee (p \Rightarrow \neg q)$ *a true statement?*

This would be very difficult to answer without truth tables, but with them we can do it step by step... First construct the truth table for $\neg q$:

p	q	$\neg q$
T	T	F
T	F	T
F	T	F
F	F	T

Now compare the columns p and $\neg q$ to add the column for $p \Rightarrow \neg q$:

p	q	$\neg q$	$p \Rightarrow \neg q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

Now compare p and $p \Rightarrow \neg q$ to construct the column for $p \vee (p \Rightarrow \neg q)$:

p	q	$\neg q$	$p \Rightarrow \neg q$	$p \vee (p \Rightarrow \neg q)$
T	T	F	F	T
T	F	T	T	T
F	T	F	T	T
F	F	T	T	T

So we have shown that $p \vee (p \Rightarrow \neg q)$ is true for all possible combinations of truth inputs. This is an example of a tautology!

Formally, a *tautology* is a compound statement which is true for all combinations of truth inputs.

Conversely, a compound statement which is false for all combinations of truth inputs is called a *contradiction*. For example. $p \wedge \neg p$ is a contradiction:

p	$\neg p$	$p \wedge \neg p$
T	F	F
T	F	F
F	T	F
F	T	F

Truth tables are also an excellent way of showing that two statements 'mean the same thing'; two statements are *equivalent* if they have the same truth tables. We say p "is equivalent to" q, and write $p \equiv q$.

2.4 Important Equivalencies

When studying logic, you will almost certainly meet the De Morgan's laws and the distributive laws. These are incredibly helpful for manipulating compound statements, and are well worth remembering!

De Morgan's laws:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q,$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Distributive laws:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r),$$
$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

These equivalencies can be proven using truth tables. For example, to prove the first of De Morgan's laws we write out the following truth table,

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q))$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

and then notice that $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q))$ have the same truth tables. Hence they are equivalent.

The other laws can be proven in the same way, so you may want to try it for yourself.

2.5 Quantifiers

There are two very important quantifiers which you will use over and over again throughout your degree, and will no doubt grow to love. (These are some of those wonderful symbols that save you loads of time when writing out proofs!)

- The *universal quantifier*, denoted $\forall$, means "for all".
- The *existential quantifier*, denoted $\exists$, means "there exists".

So if ϕ is a predicate, then $\forall x\ \phi(x)$ means "for all x, $\phi(x)$ is T", and $\exists x\ \phi(x)$ means "there exists an x such that $\phi(x)$ is T". Note that with $\exists$, a "such that" is included so that the statement makes grammatical sense. For this reason, $\exists x\ \phi(x)$ is often written $\exists x$ s.t. $\phi(x)$, where "s.t." stands for "such that".

Where a statement contains more than one quantifier, the order of $\forall$ and $exists$ cannot be interchanged, but two consecutive $\exists$'s and $\forall$'s can be. So

$$\begin{aligned}\forall x\ \forall y\ \phi(x,y) &\equiv \forall y\ \forall x\ \phi(x,y)\\ \exists x\ \exists y\ \phi(x,y) &\equiv \exists y\ \exists x\ \phi(x,y)\\ \forall x\ \exists y\ \phi(x,y) &\not\equiv \exists y\ \forall x\ \phi(x,y)\end{aligned}$$

2.6 Negation of Quantifiers

If you want to find the negation of a statement, then the following two results will be very helpful. Again worth remembering!

$$\begin{aligned}\neg(\forall x\ \phi(x)) &\Leftrightarrow \exists x \text{ s.t.} \neg\phi(x)\\ \neg(\exists x\ \phi(x)) &\Leftrightarrow \forall x\ \neg\phi(x)\end{aligned}$$

Therefore, to negate a statement, we simply 'work the negation through the statement' using the two results above.

Example *Find the negation of* $\forall x\ \exists y$ *s.t.* $\phi(x,y)$.

So we write:

$$\begin{aligned}\neg(\forall x\ \exists y \text{ s.t. } \phi(x,y)) &\Leftrightarrow \exists x \text{ s.t. } \neg(\exists y \text{ s.t. } \phi(x,y))\\ &\Leftrightarrow \exists x \text{ s.t. } \forall y\ \neg\phi(x,y)\end{aligned}$$

Note that $\nexists$ is often used in place of $\neg\exists$, but $\not\forall$ is hardly ever used (because it looks daft!).

3 Sets

3.1 Definitions and Notation

A *set* is simply a collection of objects. These objects are called the *elements of the set*, and can be anything at all! For example, we could have a set containing the days of the week, or a set containing really good types of cake. However, as mathematicians, we will be mostly concerned with... Well, more mathematical sets, such as sets of numbers.

A set is usually denoted by a capital letter, and an element of the set is denoted by a lowercase letter. Hence, if x is an element of A, we write $x \in A$. "$\in$" is the symbol mathematicians like to use to mean "is an element of". If y is not an element of A, we write $y \notin A$. Sets are sometimes denoted by listing all of the elements in curly brackets, for example, if A was the set of days of the week, we could write

$$A = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\} .$$

Many sets, however, are infinite and so it is not possible to list all the elements like this. Consider the set B of prime numbers, which can be written in either of the following forms:

$$B = \{2, 3, 5, 7, 11, ...\}$$
$$B = \{x \mid x \text{ is a prime number}\}$$

Here we have used the symbol "|", which is a shorthand way of writing "such that"; so the above example would read "B has elements x, such that x is a prime number". Now we can write $47 \in B$ and $4 \notin B$.

Here are some useful things to remember about sets:

- Two sets are equal if, and only if, they contain the same elements.
- Repeated elements in a set are ignored, for example, $\{1, 1, 1, 2\} = \{1, 2\}$.
- The order in which the elements are listed is not important, $\{2, 1\} = \{1, 2\}$.

The *size* of a set A is the number of distinct element in A; formally, this is known as the *cardinality* of the set and is denoted $|A|$. So, $|\{1, 2\}| = 2$, whereas the cardinality of $B = \{x \mid x \text{ is a prime number}\}$ is infinite.

The set with cardinality 0 (that is, the set with no elements) is called the *empty set*, denoted by $\emptyset$. The 'opposite' of $\emptyset$ is the *universal set*, denoted Ω or $\mathcal{U}$; this is the set of all elements that could be involved in the problem under consideration.

3.2 Relationships Between Sets

As inquisitive mathematicians, we are often interested in how sets interact with each other. Here are some important definitions:

- We say that A is a *subset* of B, written $A \subseteq B$, if all elements of A are also elements of B, that is:

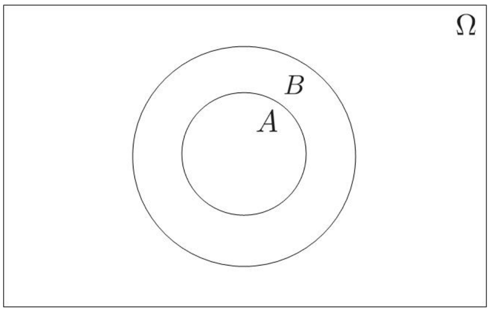
$$A \subseteq B \Leftrightarrow x \in A \Rightarrow x \in B$$

  Note that this includes the case where $A = B$.

- We say that A is a *strict subset* of B, written $A \subset B$, if$A \subseteq B$ and $A \neq B$

- A is a *proper subset* of B if $A \subset B$ (so $A \neq B$) and $A \neq \emptyset$.

- Venn diagrams are a very effective way of representing relations between sets. For example:

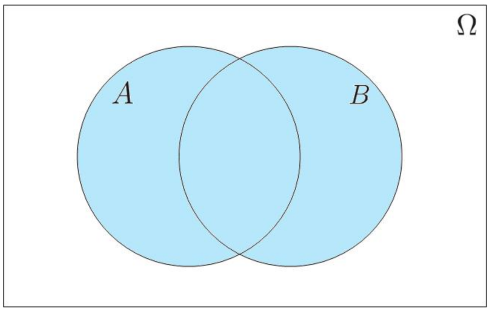


$A \subset B$

- The *union* of A and B, denoted $A \cup B$, is the set of elements which which are in A, or in B, or in both A and B. That is:

$$A \cup B = \{x \in \Omega \mid x \in A \vee x \in B\},$$

  where $\vee$ denotes 'or' (see Chapter 2).

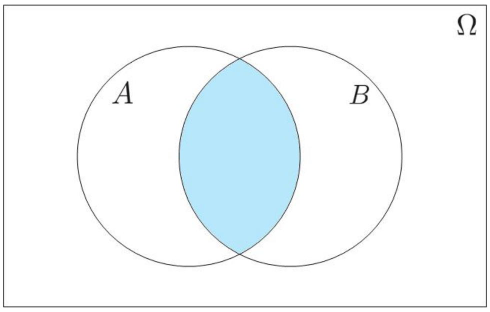


$A \cup B$

- The *intersection* of A and B, denoted $A \cap B$, is the set of elements which which are in both A and B. That is:

$$A \cap B = \{x \in \Omega \mid x \in A \wedge x \in B\},$$

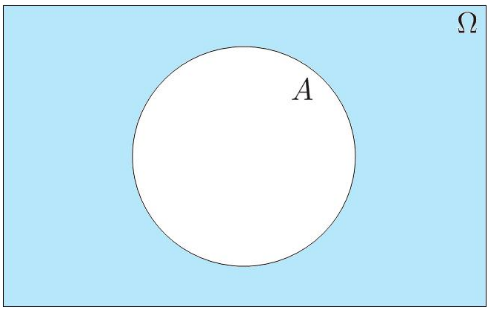
  where $\wedge$ denotes 'and'.



$A \cap B$

- The *complement* of A, written A^c, is the set of elements in he universal set which are not in A. So,

$$A^c = \{x \in \Omega \mid x \notin A\}.$$

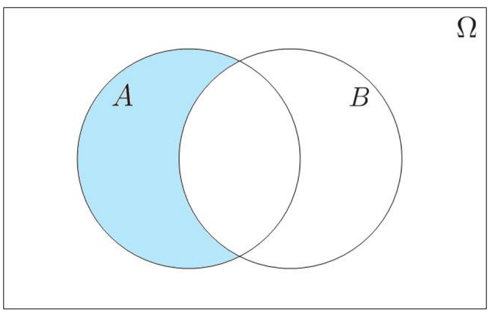


A^c

- The *set difference* of A and B, denoted $A \setminus B$, is the set of elements in A which are not in B. There are two ways of expressing this mathematically:

$$A \setminus B = \{x \in A \mid x \notin B\}, \text{or}$$
$$A \setminus B = \{x \in \Omega \mid x \in A \wedge x \notin B\}.$$

$A \setminus B$ is often called "*A less B*".



$A \setminus B$

3.3 Laws of Sets

Just like the predicates p and q in Chapter 2, De Morgan's laws and the distributive laws also apply to sets:

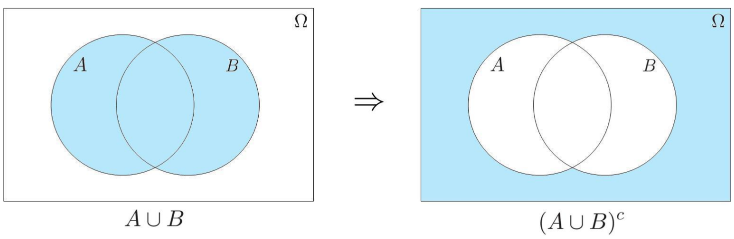
De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c$$

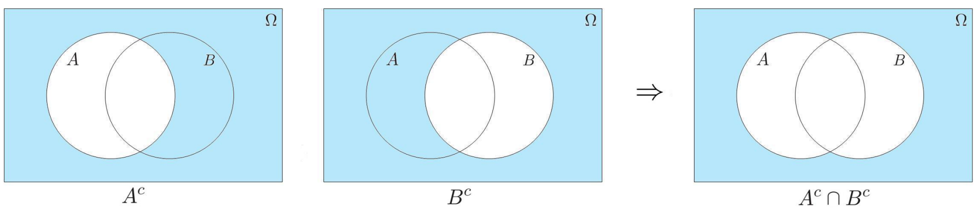
Distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

It is easy to see that these equalities hold using Venn diagrams. For example, for the first of De Morgan's laws, we can represent $(A \cup B)^c$ as follows:



And now if we look at $A^c \cap B^c$ ....



The Venn diagrams for $(A \cup B)^c$ and $A^c \cap B^c$ are the same, so we conclude that $(A \cup B)^c = A^c \cap B^c$. Ta-da!

Warning: Drawing Venn diagrams can be very useful to help us see that the laws are true, but they do **not** suffice to prove the results . We will discuss rigourous proofs in Chapter 5.

Try to represent the remaining De Morgan's law and distributive laws as Venn diagrams, to convince yourself that they are true.

3.4 Axioms of Set Theory

An *axiom* is a property that is universally accepted as true; they are the fundamental facts from which all other properties are derived (and are necessarily few and far between!). The most important axioms of set theory are given below:

- $(A^c)^c = A$
- $A \cup A^c = \Omega$
- $A \cap A^c = \emptyset$

We also accept commutativity and associativity of $\cap$ and $\cup$ as axioms:

Commutativity

$$\begin{aligned} A \cup B &= B \cup A, \\ A \cap B &= B \cap A \end{aligned}$$

Associativity

$$\begin{aligned} (A \cap B) \cap C &= A \cap (B \cap C), \\ (A \cup B) \cup C &= A \cup (B \cup C) \end{aligned}$$

3.5 Sets of Sets

A set can, and often will, contain other sets; that is, as set could be an element of another set. For example, {{1,2}, {3,4}} is the set with elements {1,2} and {3,4}.

An important example of a 'set of sets' is the *power set* of a set X. Denoted $\mathcal{P}(X)$, the power set of X is the set of all subsets of X. For example, for $X = \{1, 2, 3\}$:

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Note that $\emptyset$ and the entire set are subsets of all sets, and so are always in the power set.

4 Numbers

Many different sets of numbers are of interest to mathematicians. Whilst there are some very obscure sets lurking out there (such as the hyperreals, surreals and p-adic numbers), mathematicians usually concern themselves primarily with:

- $\mathbb{N}$, the *natural numbers*;
- $\mathbb{Z}$, the *integers*;
- $\mathbb{Q}$, the *rational numbers*;
- $\mathbb{R}$, the *real numbers*; and,
- $\mathbb{C}$, the *complex numbers*.

It is likely that you are already familiar with most, or even all, of these sets of numbers. However, it is possible that you have not yet given much thought to their individual properties. So here is a lightning tour of some of the more interesting ones.

4.1 The Natural Numbers, $\mathbb{N}$

The set of natural numbers, denoted by $\mathbb{N}$, can be defined:

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, ...\}$$

More formally (but not completely formally!), we can construct the natural numbers as follows:

> *Apart from 1, each natural number is the successor of exactly one other natural number. For example. 2 is the successor of 1, 3 is the successor of 2, and so on. 1 is defined to be the unique natural number which is not the successor of any other natural number.*

It is worth noticing that:

- there is no largest element of the natural numbers, since if you had the "largest" natural number, you could always add 1 to create a larger one.
- $\mathbb{N}$ is closed under addition and multiplication, but nothing else.

By "closed under $\circ$", we mean that you will not leave the set performing the operation $\circ$ on the elements. For example, $\mathbb{N}$ is closed under $+$, since adding any natural number to any other natural number, will always give another natural number. However, $\mathbb{N}$ is not closed under $-$, since subtracting one natural number from another may mean that you leave the natural numbers; for example:

$$5 - 6 = -1.$$

Here, 5 and 6 are both natural numbers, but you may have a sneaky feeling that -1 is not...

4.2 The Integers, $\mathbb{Z}$

The set of integers, $\mathbb{Z}$, can be written:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

The integers consist of all "whole" numbers; they are made up of the natural numbers, together with the negatives of the natural numbers and zero. Therefore, the natural numbers are a subset of the integers, that is to say, $\mathbb{N} \subset \mathbb{Z}$, and so $\mathbb{N}$ can be written as:

$$\mathbb{N} = \{z \in \mathbb{Z} \mid z > 0\}.$$

Here are some properties of $\mathbb{Z}$:

- $\mathbb{Z}$ contains -1, which is used to construct the negative numbers, much like $\mathbb{N}$ is constructed with 1.
- like $\mathbb{N}$, $\mathbb{Z}$ is closed under addition and multiplication, but $\mathbb{Z}$ also allows subtraction.

4.3 The Rational Numbers, $\mathbb{Q}$

The rational numbers, $\mathbb{Q}$, are the numbers that can be expressed as fractions using integers, that is to say:

$$\mathbb{Q}= \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$$

Some properties of $\mathbb{Q}$:

- $\mathbb{Q}$ contains $\mathbb{Z}$, since

$$\mathbb{Z} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0, a = nb, n \in \mathbb{Z}\} \subset \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\} = \mathbb{Q}$$

- Like $\mathbb{Z}$, $\mathbb{Q}$ is also closed under addition, subtraction and multiplication; however, $\mathbb{Q}$ also allows division by non-zero integers.
- Every rational number can be written as a finite or recurring decimal.

4.4 The Real Numbers, $\mathbb{R}$

The real numbers, $\mathbb{R}$, are a bit more difficult to define, formally:

> *The real numbers are the smallest set containing all limits of convergent sequences of rational numbers.*

That's a bit scary sounding really, but have no fear! We discuss limits and convergence later in Chapter 7... Less formally, the rational numbers are the numbers that can be written as decimals (including infinite decimals, such as $\pi = 3.14159265...$). Practically, real numbers measure continuous data such as length at time.

Note that all fractions can be written as decimals, so $\mathbb{Q} \subset \mathbb{R}$. The set of *irrational numbers*, that is to say, numbers which cannot be expressed as a fraction of integers (or a finite or recurring decimal), is denoted by $\mathbb{R}\backslash\mathbb{Q}$, since any real number that is not rational is (obviously!) irrational.

4.5 The Complex Numbers, $\mathbb{C}$

Most of you will already be familiar with the concept of a complex, or "imaginary", number. But for those of you who are not, here are the basics.

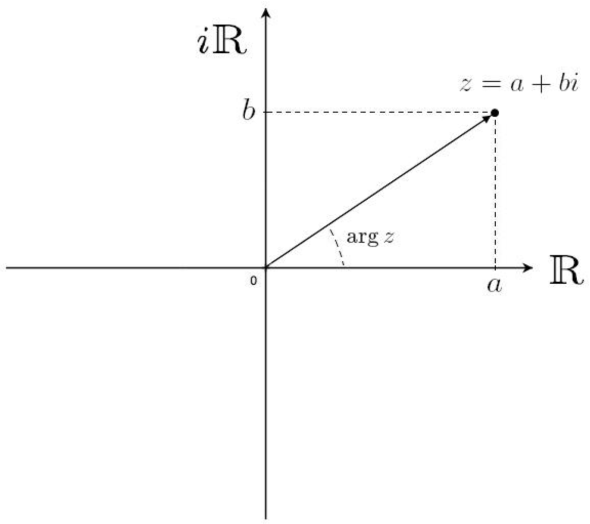
The complex numbers, $\mathbb{C}$, are defined to be:

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}, i^2 = -1\}$$

So basically, we can define i to be the solution of $x^2 + 1 = 0$, and use this to generate a whole new set of numbers of the form $a + ib$. So the real part of $a + ib$ is a, and the imaginary part is ib. These numbers will be a regular feature of your degree!

The exciting properties of $\mathbb{C}$ include:

- $\mathbb{C}$ contains $\mathbb{R}$, since if we set $b = 0$, there is no imaginary part of $a + ib$.
- Complex numbers are often represented graphically on something called an *Argand diagram*:



- Unlike $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, $\mathbb{C}$ has no natural ordering. That is to say, we cannot say that one complex number precedes another, because this makes no sense! (One way to think about this is to consider trying to order the points in a plane...)
- Like $\mathbb{R}$, $\mathbb{C}$ is closed under addition, subtraction, multiplication and division, where addition and multiplication are defined exactly as you would expect them to be:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$
$$(a + ib)(c + id) = ac - bd + i(ad + bc)$$
$$\frac{a + ib}{c + id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

  Notice that, when multiplying, we are just expanding the brackets and then using the fact that $i^2 = -1$. Not too difficult!

- Every complex number, $z = x + iy$, has a *complex conjugate*, $\overline{z} = x - iy$. This is useful because $z\overline{z}$ is always a real number, since $(x + iy)(x - iy) = x^2 + y^2$.

We can also express complex numbers in what is know as the *phasor form*. If we let $z = x + iy$ be our favourite complex number, then we can see from our beautiful diagram that

$$z = |z|(cos\theta + isin\theta)$$

Now, using the *Euler identity*: $e^{ix} = cosx + isinx$, we have that

$$z = |z|e^{i\theta}.$$

$|z|$ is commonly called the *complex modulus* of z, and θ the *complex argument* of z (sometimes denoted $arg(z)$). We can compute $|z|$ and θ explicitly using the following formulae:

$$|z| = \sqrt{z\overline{z}}$$
$$\theta = arctan\left(\frac{y}{x}\right)$$

It also easy to compute the real and imaginary part of z, denoted $Re(z)$ and $Im(z)$ respectively:

$$Re(z) = \frac{1}{2}(z + \overline{z})$$
$$Im(z) = \frac{1}{2}i(\overline{z} - z)$$

Finally, before we move on to exciting proofs, there's one more really useful result you should know about complex numbers. *De Moivre's identity* can be used to find z^n, where $n \in \mathbb{R}$:

$$z^n = |z|^n[cos(n\theta) + isin(n\theta)]$$

5 Proofs

Suppose we wanted to prove the statement "*all birds have beaks*" There are three ways we could do this:

1. We could find every bird and show that it has a beak. This is an example of a *direct proof*.
2. We could suppose that the statement is not true, taking the negation of the statement to get "*there exists a bird which does not have a beak*" (recall from Chapter 2 that "for all" changes to "there exists" after negation). We would then show that this 'bird' was not a bird at all. So we'd have a bird which wasn't a bird. Which is mad, so the assumption we made must have been wrong! We therefore can't find a bird without a beak, so all birds must have beaks. This is an example of a *proof by contradiction*
3. We could show that anything that doesn't have a beak isn't a bird. So any bird we find isn't in the set of things which don't have a beak, and so must have a beak. This is an example of a *proof by contrapositive*.

So let's have a look at these three methods of proof in slightly more detail.

5.1 Direct Proof

Direct proof is proving the statements 'as it is'; that is, if we want to prove $p \Rightarrow q$, we suppose that p is true and prove that q must also be true.

Let's look at an example of a direct proof.

Definition a *divides* b means $\exists k \in \mathbb{N}$ s.t. $b = ak$ (i.e. b is a multiple of a).

Theorem If a divides b and b divides c, then a divides c.

Proof Suppose a divides b and b dives c. Then $\exists k_1, k_2 \in \mathbb{N}$ s.t. $b = ak_1$ and $c = bk_2$. Substituting $b = ak_1$ into $c = bk_2$, we get that $c = ak_1k_2 \Rightarrow c = ak$, where $k = k_1k_2 \in \mathbb{N}$. Hence a divides c.

When proving something directly, we often use a method called *proof by induction*, which you may have met already. Proof by induction is a method of "proving patterns", which relies on the construction of the natural numbers (recall from Chapter 3 that, apart from 1, each natural number is the successor of exactly one other natural number).

So let $P(n)$ be a proposition about $n \in \mathbb{N}$. Then if we can show that:

1. $P(1)$ is true, and
2. if $P(r)$ is true, then $P(r+1)$ is true $\forall r \in \mathbb{N}$,

we have proved that $P(n)$ is true for every $n \in \mathbb{N}$. For example, we can use induction to prove the following theorem (which you might recognise!)...

Theorem

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Proof Let $P(n)$ be the statement $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, that is, $1+2+3+...+n = \frac{1}{2}n(n+1)$.

1. $P(1)$ says that $1 = \frac{1}{2}1(1+1) = 1$. This is what we would hope for! So the result is true for $n = 1$.

2. Suppose the result is true for $n = k$, that is, $P(k)$ is true and $1+2+3+...+k = \frac{1}{2}k(k+1)$. Now we want to show that $P(k = 1)$ is true...

$$\begin{aligned}
P(k+1) &= \sum_{r=1}^{k+1} r \\
&= \sum_{r=1}^{k} r + (k+1) \\
&= \frac{1}{2}k(k+1) + k + 1 \\
&= \frac{1}{2}(k^2+k) + (k+1) \\
&= \frac{1}{2}(k^2+k) + \frac{1}{2}2(k+1) \\
&= \frac{1}{2}(k^2+k+2k+2) \\
&= \frac{1}{2}(k^2+3k+2) \\
&= \frac{1}{2}(k+1)(k+2)
\end{aligned}$$

   So the result is true for $P(k+1)$ and we have finished the proof.

5.2 Proof by Contradiction

Another way to prove $p \Rightarrow q$ is to suppose that the statement is not true, i.e. $\neg(p \Rightarrow q)$, and then find some sort of contradiction which shows that p is not true. In terms of logical statements, we are saying that:

$$\neg(p \Rightarrow q) \Rightarrow \neg p \;\equiv\; p \Rightarrow q$$

We can show that this is true using truth tables (see Chapter 1):

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg p$	$\neg(p \Rightarrow q) \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	F	T	T

We can now see that $p \Rightarrow q$ and $\neg(p \Rightarrow q) \Rightarrow \neg p$ have the same truth tables, and are therefore equivalent.

Let's look at an example of a proof by contradiction...

Theorem There is no greatest even integer.

Proof Suppose this statement is not true, and that there is a greatest even integer, which we will imaginatively call N. Now we seek a contradiction.

If N is the greatest even integer, then for every even integer n, $N \geq n$. Let $M = N + 2$. Then M is an even integer, since it is the sum of two even integers. Also, $M > N$. So M is an even integer greater than N. This contradicts our initial supposition (that there is a greatest even integer N), therefore there is no greatest even integer.

5.3 Proof by Contrapositive

The third way to prove $p \Rightarrow q$ is to instead prove $\neg q \Rightarrow \neg p$. That is, instead of showing p true $\Rightarrow q$ true, we show p false $\Rightarrow q$ false.

We can see that $p \Rightarrow q \equiv (\neg q \Rightarrow \neg p)$ by once again using truth tables:

p	q	$p \Rightarrow q$	$\neg q)$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Again, we see that $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$ have the same truth tables and are therefore equivalent.

Let's look at an example...

Definition An integer x is *even* if $\exists k \in \mathbb{Z}$ s.t. $x = 2k$.

Definition An integer y is *odd* if $\exists m \in \mathbb{Z}$ s.t. $y = 2m + 1$.

Definition Two integers are said to have the same *parity* if both integers are odd or if both integers are even.

Theorem If x and y are two integers for which $x+y$ is even, then x and y have the same parity.

Proof The contrapositive version of this theorem is "if x and y are two integers with different parity, then $x + y$ is odd". So assume x and y have opposite parity (one is even and the other is odd), so one equals $2k$ for some $k \in \mathbb{Z}$ and the other equals $2m + 1$ for some $m \in \mathbb{Z}$. Then:

$$
\begin{aligned}
x + y &= 2k + 2m + 1 \\
&= 2(k + m) + 1 \\
&= 2N + 1
\end{aligned}
$$

where $N = k + m \in \mathbb{Z}$. So $x + y$ is odd by definition.

Up to now, we have just been trying to prove a theorem is true. However, showing that it is not true is often much simpler, since all you need to do is find one example for which the theorem doesn't hold. This is called a *counter-example*, and you'll grow to love them in time! Let's look at an example...

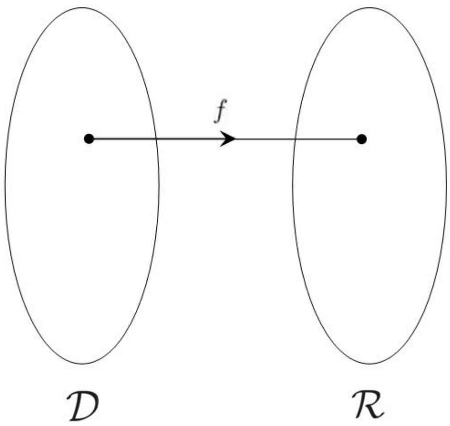
Theorem If $a + b$ is an integer, then a and b are integers.

This must be false, since we can take $a = 1.5$ and $b = 2.5$, then $a + b = 4$, which is an integer. But a and b are not integers, and so we have found a counter-example which proves the theorem is false.

6 Functions

6.1 A Few Important Definitions

A *map* is a rule from one set to another. If the map f takes things in $\mathcal{D}$ to things in $\mathcal{R}$, we write $f : \mathcal{D} \to \mathcal{R}$ and say that "f is a map from $\mathcal{D}$ to $\mathcal{R}$".



Definition A map f from $\mathcal{D}$ to $\mathcal{R}$ is a *function* if:

1. f is defined for every element of $\mathcal{D}$.
2. For every $x \in \mathcal{D}$, there is a single point y in $\mathcal{R}$ s.t. $y = f(x)$.

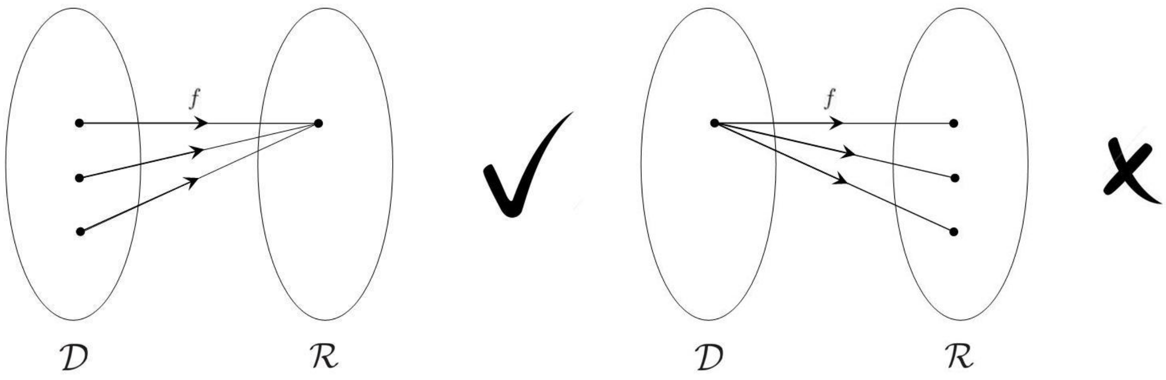
That is, each point in $\mathcal{D}$ corresponds to a **unique** point in $\mathcal{R}$. $\mathcal{D}$ is called the *domain* of the function, $\mathcal{R}$ is called the *codomain* of the function.

Examples

- Let $\mathcal{D} = (0, 1)$, $\mathcal{R} = \mathbb{R}$, $f(x) = \frac{1}{x}$. Then $f : \mathcal{D} \to \mathcal{R}$ is a function, since it is defined $\forall x$ s.t. $0 < x < 1$.
- Now let $\mathcal{D} = [0, 1]$, $\mathcal{R} = \mathbb{R}$, $f(x) = \frac{1}{x}$. Then $f : \mathcal{D} \to \mathcal{R}$ is not a function, since $f(x)$ is not defined for $0 \leq x \leq 1$, because $\frac{1}{0}$ is undefined.
- If $\mathcal{D} = \mathbb{R}$, $\mathcal{R} = \mathbb{R}$, $f(x) = \sqrt{x}$. Then $f : \mathcal{D} \to \mathcal{R}$ is not a function, since each $x \in \mathcal{D}$ does not correspond to a unique $y \in \mathcal{R}$. For example, ± 1 are both equal to $\sqrt{1}$. However, if we set $\mathcal{R} = \mathbb{R}^+$, the positive reals, then $f(x)$ is a function.

Note that it is the points in the codomain that must be unique, but the points in the domain need not be. So $f(x) = x^2$ on $\mathcal{D} = \mathbb{R}$, $\mathcal{R} = \mathbb{R}$, is a function even though $x = -1$ and $x = 1$ both give the same value (1) in $\mathbb{R}$.

So functions can be "many-to-one", but they can't be "one-to-many":



A good way to remember which way is a function and which way isn't is to note that if you're driving and have had "one too many", then you can't function!

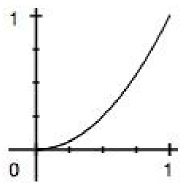
Definition Suppose f is a function on $\mathcal{D}$, then f is *increasing* if

$$x < y \Rightarrow f(x) \leq f(y).$$

f is *strictly increasing* if

$$x < y \Rightarrow f(x) < f(y).$$

Example $f(x) = x^2$ on $[0, 1]$ is strictly increasing.

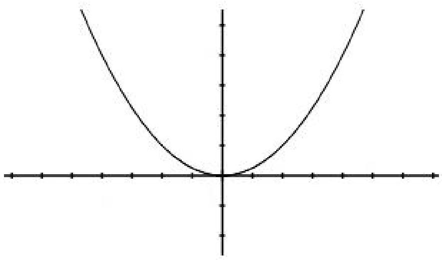


Proof Suppose $x < y$. $f(y) = y^2$, so:

$$\begin{aligned} f(y) - f(x) &= y^2 - x^2 \\ &= (y - x)(y + x) \\ &> 0 \end{aligned}$$

since $x < y \Rightarrow y - x > 0$, so $x + y > 0$. Hence $y^2 > x^2$, and so $f(y) > f(x)$.

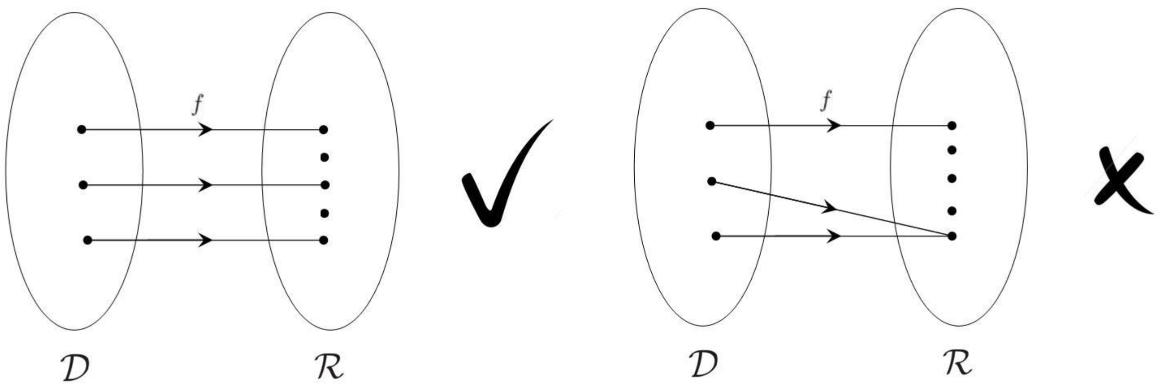
Example $f(x) = x^2$ on $\mathbb{R}$ is not increasing; for example, $-1 < 0$, but $f(-1) = 1 > 0 = f(0)$. Note that the proof we used in the previous example now fails because $x, y \in \mathbb{R} \not\Rightarrow x + y > 0$.



Similarly, a function is *decreasing* if $x < y \Rightarrow f(x) \geq f(y)$, and *strictly decreasing* if $x < y \Rightarrow f(x) > f(y)$.

Definition A function is said to be a *(strictly) monotone* function if it is either (strictly) increasing or (strictly) decreasing.

Definition A function is said to be *injective* if it is one-to-one (as opposed to many-to-one), so it never maps distinct elements of its domain to the same element of its codomain. More formally, f is injective if $\forall x, y \in \mathcal{D}, f(x) = f(y) \Rightarrow x = y$.

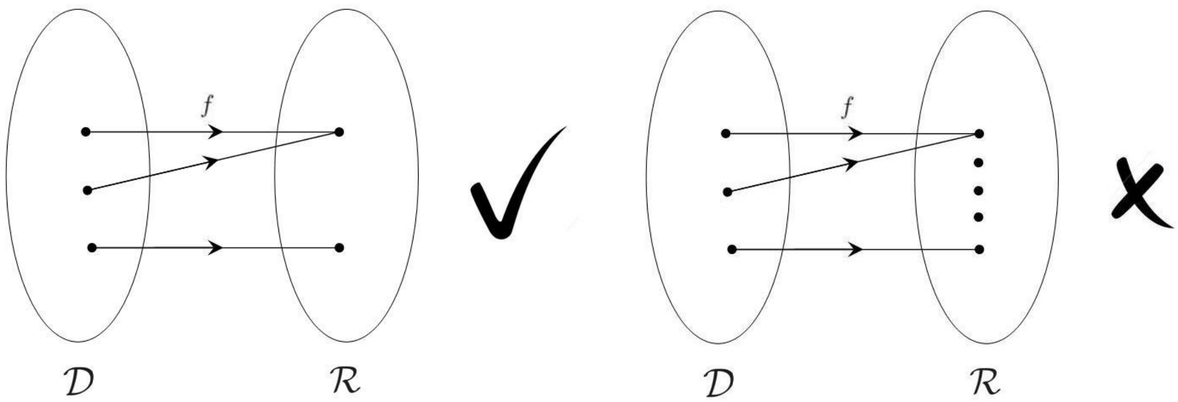


A good way to visualise injectivity...

Examples

- $f(x) = 2x$ from $\mathbb{R} \to \mathbb{R}$ is injective. If $f(x) = f(y)$ then $2x = 2y \Rightarrow x = y$.
- $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$ is not injective. If $f(x) = f(y)$ then $x^2 = y^2 \not\Rightarrow x = y$, since x could equal $-y$. For example, $f(1) = f(-1)$, but $1 \neq -1$.
- $f(x) = x^2$ from $\mathbb{R}^+ \to \mathbb{R}$ is injective.

Definition A function is *surjective* if for every y in the codomain, there is at least one x in the domain s.t. $f(x) = y$. In other words, everything in the $\mathcal{R}$ gets mapped to by something in $\mathcal{D}$.



And a good way to visualise surjectivity...

Examples

- $f(x) = 2x$ from $\mathbb{R} \to \mathbb{R}$ is surjective, since $\forall y \in \mathbb{R}$, there is an $x \in \mathbb{R}$ s.t. $f(x) = y$, namely $x = \frac{y}{2}$

- $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$ is not surjective since, for example, there is no $x \in \mathbb{R}$ s.t. $f(x) = -1$.
- $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}^+$ is surjective. Note that for injectivity we needed to restrict the domain, and for surjectivity we restricted the codomain.

Definition A function is *bijective* if it is injective and surjective.

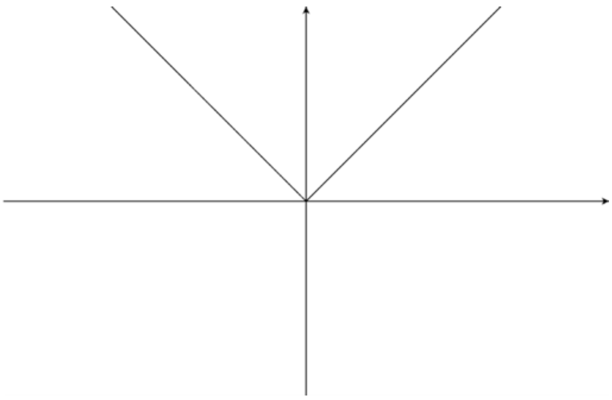
Example $f(x) = 2x$ from $\mathbb{R} \to \mathbb{R}$ is bijective.

6.2 The (Extremely Important) Modulus Function

The *modulus* (or absolute value) of a number is a measure of its size. Formally, the modulus of x, denoted $|x| = max\{x, -x\}$. Equivalently,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

If we plot this, we get a graph that looks a bit like this (actually it looks exactly like this):



$f(x) = |x|$

For example, $|3| = 3$, $|-14| = 14$, and $|0| = 0$.

$|x|$ is a function with domain $\mathcal{D} = \mathbb{R}$ and codomain $\mathcal{R} = \mathbb{R}^+$. Here are a few other useful facts about $|x|$:

- $x \leq |x|$ since $x \leq max\{x, -x\}$.
- $-x \leq |x|$ since $-x \leq max\{x, -x\}$.
- $|x| = |-x|$, since

$$\begin{aligned} |-x| &= max\{-x, -(-x)\} \\ &= max\{-x, x\} \\ &= max\{x, -x\} \\ &= |x| \end{aligned}$$

Theorem $|x| = \sqrt{x^2}$

Proof If $x \geq 0$, then $\sqrt{x^2} = x$.

If $x < 0$, then $x^2 = (-x)^2$, so $\sqrt{x^2} = \sqrt{(-x^2)} = -x$. Hence

$$\sqrt{x^2} = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} = |x|$$

This theorem can be used to help prove other statements about the modulus function. Let's see an example...

Theorem $|ab| = |a||b|$

Proof $|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b|$.

Note that this can be proven directly from the definition of the modulus function, but it is longer and requires a 'case-by-case' approach, looking at each combination of $\pm a$ and $\pm b$, which is very tedious!

The following is a very important result which you will use a lot in any analysis courses you take:

The Triangle Inequality $|x + y| \leq |x| + |y|$

Proof Since $x \leq |x|$ and $y \leq |y|$, we have $x + y \leq |x| + |y|$.

Similarly, since $-x \leq |x|$ and $-y \leq |y|$, we have $-(x + y) = -x - y \leq |x| + |y|$. Hence,

$$\begin{aligned} |x + y| &= max\{x + y, -(x + y)\} \\ &\leq |x| + |y| \end{aligned}$$

because we have just shown that both $x + y$ and $-(x + y)$ are less than or equal to $|x| + |y|$.

The following result follows straight from the triangle inequality:

The Reverse Triangle Inequality $||x| - |y|| \leq |x - y|$

Exercise: Before you look at the proof below, try to prove the reverse triangle inequality using the triangle inequality. Here's a hint: show that $|x| - |y| \leq |x - y|$ and $-(|x| + |y|) \leq |x - y|$...

Proof

$$\begin{aligned} |x| &= |y + (x - y)| \\ &\leq |y| + |x - y| \qquad \text{(using the triangle inequality)} \\ \Rightarrow |x| - |y| &\leq |x - y| \end{aligned}$$

Similarly,

$$\begin{aligned} |y| &= |x + (y - x)| \\ &\leq |x| + |y - x| && \text{(the triangle inequality again)} \\ \Rightarrow |y| - |x| &\leq |y - x| \\ &= |-(y - x)| \\ &= |x - y| \end{aligned}$$

Hence $||x| - |y|| \leq |x - y|$, and we are done.

7 Limits of Functions

7.1 Definitions and the Algebra of Limits

There are many functions that aren't defined for certain values, for example $\frac{1}{x}$ isn't defined at 0. So we can't evaluate $\frac{1}{x}$ at 0, but we can see what happens as x gets closer and closer to 0. The correct way to say this is "as x *tends* to 0", written $x \to 0$.

Example

$$f(x) = \frac{x^3 - 1}{x - 1} \qquad\qquad x \neq 1$$

$f(x)$ is defined for all real numbers except for 1, so we can't evaluate $f(x)$ at $x = 1$. However, we can look at what happens as $x \to 1$.

As x tends to 1 from below:

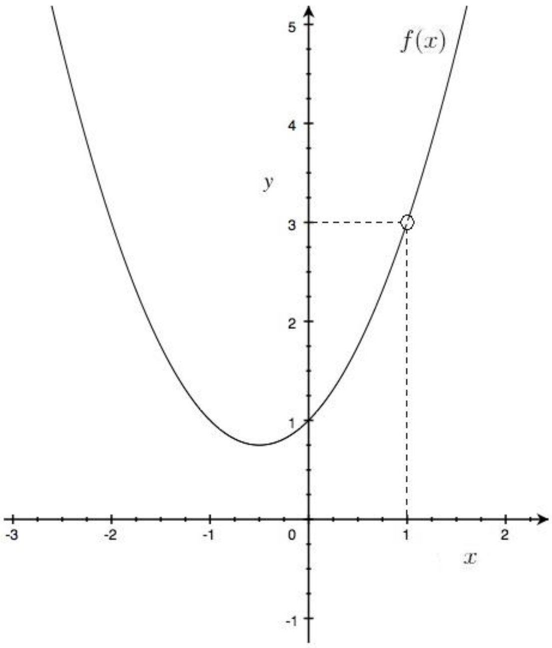
x	$f(x)$
0.8	2.44
0.9	2.71
0.95	2.8525
0.97	2.9109
0.98	2.9404
0.99	2.9701
$\vdots$	$\vdots$
0.999	2.997

As x tends to 1 from above:

x	$f(x)$
1.2	3.64
1.1	3.31
1.05	3.1525
1.03	3.0909
1.02	3.0604
1.01	3.0301
$\vdots$	$\vdots$
1.001	3.003

We can see that $f(x)$ appears to be tending to 3 as $x \to 1$. This can also be seen if we look at the graph of $f(x)$ on $\mathbb{R}\backslash\{1\}$.

Also note that on $\mathbb{R}\backslash\{1\}$, $f(x) = \frac{x^3-1}{x-1} = x^2 + x + 1$. We therefore make the guess that $f(x)$ tends to 3 as $x \to 1$. If this is true, we say that $f(x)$ has *limit* $l = 3$ at $x = 1$.



$f(x) = \frac{x^3-1}{x-1}$

Definition Let $f(x)$ be a function defined on $\mathbb{R}\backslash\{a\}$. We say that $f(x)$ has a limit L at the point $x = a$ if for all sufficiently small $\varepsilon > 0$ there exists $a > 0$ s.t.:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

and we write $\lim\limits_{x \to a} f(x) = L$.

Basically, $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ is saying that if a is very close to x then $f(x)$ is very close to L.

We will now use this definition to prove rigourously that $f(x)$ has limit $L = 3$ at $x = 1$:

We suppose that $0 < |x - a| < \delta$ for some $\delta > 0$ and look at $|f(x) - L|$, then show that it is less than ε, $\forall \varepsilon > 0$.

$$\begin{aligned}
|f(x) - L| &= |(x^2 + x + 1) - 3| \\
&= |x^2 + x - 2| \\
&= |(x-1) + (x^2 - 1)| \\
&\leq |x-1| + |x^2 - 1| && \text{by the triangle inequality} \\
&\leq |x-1| + |x-1||x+1| \\
&= |x-1|(1 + |x+1|) \\
&= |x-1|(1 + |(x-1) + 2|) \\
&\leq |x-1|(1 + (x-1) + 2) && \text{by the triangle inequality} \\
&= |x-1|(3 + |x+1|) \\
\Rightarrow |f(x) - 3| &\leq |x-1|(3 + |x+1|)
\end{aligned}$$

Now suppose $\delta < 1$, so $0 < |x-1| < \delta \leq 1$. Then

$$\begin{aligned} |f(x)-3| &\leq \delta(3+|x-1|) && \text{using } 0<|x-1|<\delta \\ &\leq \delta(3+1) && \text{using } 0|x-1|<1 = 4\delta \end{aligned}$$

Therefore

$$0 < |x-1| < \delta \Rightarrow |f(x)-3| < 4\delta$$

Hence we can choose $\delta = \frac{\varepsilon}{4}$, and then $\forall \varepsilon > 0$:

$$|x-1| < \delta \Rightarrow |f(x)-3| < \varepsilon$$

So our definition holds for $x = 1$, or $\lim\limits_{x\to 1} f(x) = 3$, as required. Hooray!

Example $f(x) = x$ has limit a at $x = a$.

Proof Let $|x-a| < \delta$ and consider $|f(x)-L|$. Now $|f(x)-L| < |x-a| < \delta$, so if we set $\delta = \varepsilon$ then

$$|x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

Not so difficult!

The following theorem basically states that you can manipulate limits as if they were numbers. Which is pretty useful! You'll definitely use this one a lot...

The Algebra of Limits Suppose $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = M$, then:

1. $\lim\limits_{x\to a}(\lambda f(x) + \mu g(x)) = \lambda L + \mu M$, for $\lambda, \mu \in \mathbb{R}$
2. $\lim\limits_{x\to a}(f(x)g(x)) = LM$
3. If $M \neq 0$, $\lim\limits_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$

Proof We will prove the first part of theorem, the second and third parts are similar.

As $\lim\limits_{x\to a} f(x) = L$, then $\forall\varepsilon > 0, \exists\delta_1 > 0$ s.t.

$$0 < |x-a| < \delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2|\lambda|} \tag{1}$$

and since $\lim\limits_{x\to a} g(x) = M$, then $\forall\varepsilon > 0, \exists\delta_2 > 0$ s.t.

$$0 < |x-a| < \delta_2 \Rightarrow |g(x)-M| < \frac{\varepsilon}{2|\mu|} \tag{2}$$

by applying the definition of a limit to $f(x)$ and $g(x)$. Now consider $|(\lambda f(x) + \mu g(X) - (\lambda L + \mu M)|$, we want to show that this is less than ε...

$$\begin{aligned} |\lambda f(x) + \mu g(x) - \lambda L - \mu M| &= |\lambda(f(x)-L) + \mu(g(x)-M)| \\ &\leq |\lambda(f(x)-L)| + |\mu(g(x)-M| && \text{by the triangle inequality} \\ &\leq |\lambda||f(x)-L| + |\mu||g(x)-M| \end{aligned}$$

Now, if we choose $\delta = min\{\delta_1, \delta_2\}$, then $\delta < \delta_1$ and $\delta < \delta_2$, so $\forall x$ s.t. $|x - a| < \delta$, by (1) $|f(x) - L| < \frac{\varepsilon}{2|\lambda|}$ and by (2) $|g(x) - M| < \frac{\varepsilon}{2|\mu|}$. So

$$\begin{aligned} |(\lambda f(x) + \mu g(x)) - (\lambda L + \mu M)| &< |\lambda|\frac{\varepsilon}{2|\lambda|} + |\mu|\frac{\varepsilon}{2|\mu|} \\ &= \varepsilon \\ \Rightarrow \lim_{x\to a}(\lambda f(x) + \mu g(x)) &= \lambda L + \mu M \end{aligned}$$

as required.

Note that it follows directly from the algebra of limits that if $\lim_{x\to a} f(x) = L$, then:

1. $\lim_{x\to a} \lambda f(x) = \lambda L$ by setting $g = 0$ in the first part of the AOL, and
2. $\lim_{x\to a} (f(x))^2 = L^2$ by setting $f = g$ in the second part of the AOL.

7.2 The Closeness Theorem

We have already looked at how to show something has a limit, but what if we want to show that something does **not** have a limit? We can use something attractively named "The Closeness Theorem" to help, so let's have a look at that...

The Closeness Theorem Suppose $\lim_{x\to a} f(x) = L$, then $\forall\varepsilon > 0, \exists\delta > 0$ s.t.

$$0 < |x - a| < \delta \text{ and } 0 < |y - a| < \delta \Rightarrow |f(x) - f(y)| < 2\varepsilon$$

Proof

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - L + L - f(y)| && \text{(this is a cunning trick we use a lot in analysis)} \\ &\leq |f(x) - L| + |f(y) - L| && \text{(you guessed it - triangle inequality!)} \end{aligned}$$

Now, as $f(x) \to L$ as $x \to a$, we know that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. Also, as $f(y) \to L$ as $y \to a$, we know that if $0 < |y - a| < \delta$ then $|f(y) - L| < \varepsilon$.

Hence, if $0 < |x - a| < \delta$ and $0 < |y - a| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &< |f(x) - L| + |f(y) - L| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon \end{aligned}$$

We can now use this to show that certain functions don't have a limit. For example,

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

doesn't have a limit at $x = 1$. Even more exciting, we can prove this using the closeness theorem...

Proof

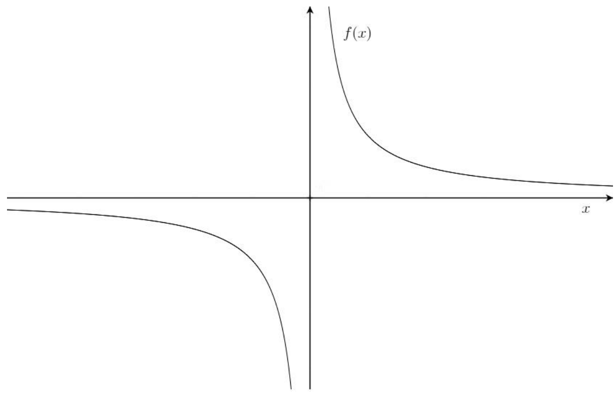
If f had a limit then, by the closeness theorem, $|f(x) - f(y)| < 2\varepsilon, \forall \varepsilon > 0$ if $0 < |x-1| < \delta$ and $0 < |y-1| < \delta$. Since this needs to hold true for all $\varepsilon > 0$, we just need to show that it fails for one specific ε to get a counter-example (see Chapter 5). So let's choose $\varepsilon = \frac{1}{4}$. Now, for the closeness theorem to hold (i.e. for f to have a limit), we should be able to find a δ for $\varepsilon = \frac{1}{4}$, s.t.

$$0 < |x-1| < \delta \text{ and } 0 < |y-1| < \delta \Rightarrow |f(x) - f(y)| < \frac{1}{2}$$

However, $\forall \delta > 0$, we can choose an $x \in \mathbb{Q}$ and a $y \notin \mathbb{Q}$ s.t. $|x-1| < \delta$ and $|y-1| < \delta$, but $f(x) = 1$ and $f(y) = 0$. So $|f(x) - f(y)| = 1 > \frac{1}{2}$.

Hence $f(x)$ doesn't satisfy the closeness theorem, and therefore we have proved that there is no limit at $x = 1$.

Example Consider $f(x) = \frac{1}{x}$, drawn below. We now have the tools to show that f does not have a limit at $x = 0$.



Proof Choose $\varepsilon = 1$. Now pick x as close as you like to 0 such that $x < \frac{1}{4}$. If we set $y = \frac{x}{2}$, the y is closer to 0 than x. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{2}{x} \right| \\ &= \left| \frac{1}{x} \right| \end{aligned}$$

But $|\frac{1}{x}| > 4 > \varepsilon$, and so the closeness theorem doesn't hold, and we have proved that f does not have a limit at $x = 0$.

7.3 Continuous Functions

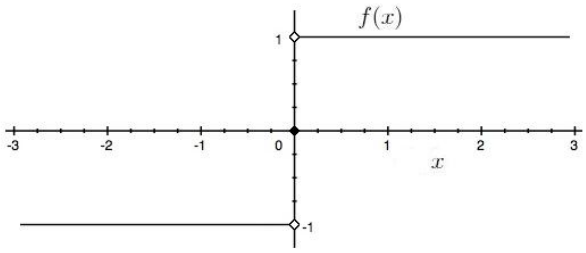
The notion of a limit is used in the definition of a "continuous function", which will probably be the most frequently used definition in your degree!

Definition A function $f(x)$ is *continuous* at the point $x = a$ if $\lim_{x \to a} f(x) = f(a)$. A function is said to be continuous if it is continuous at every point $x \in \mathcal{D}$.

Examples

- $f(x) = 1$, $f : \mathbb{R} \to \mathbb{R}$ is continuous, since $\lim_{x\to a} f(x) = 1 = f(a), \forall x \in \mathbb{R}$.
- $f(x) = x$, $f : \mathbb{R} \to \mathbb{R}$ is continuous, since $\lim_{x\to a} f(x) = a = f(a), \forall x \in \mathbb{R}$.
- $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = e^x$ are all continuous as maps from $\mathbb{R} \to \mathbb{R}$.
- What about the following function from $\mathbb{R} \to \mathbb{R}$?

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



$f(x)$ is continuous at all $x \neq 0$, but not at $x = 0$ (see the 'break' in the graph). To show that $f(x)$ is not continuous at $x = 0$ we show that $\lim_{x\to 0} f(x)$ doesn't exist, by proving that $\lim_{x\to 0} f(x) \neq f(0) = 0$.

So we wheel out the closeness theorem again...

Suppose that $0 < x < \delta \Rightarrow f(x) = 1$ and $-\delta < y < 0 \Rightarrow f(y) = -1$. Then $|x| < \delta$ and $|y| < \delta$, but $|f(x) - f(y)| = 2$. So if we pick $\varepsilon = 1$, then $|f(x) - f(y)| > \varepsilon$. So the closeness theorem fails and f has no limit at $x = 0$.

Since the definition of a continuous function only involves the definition of a limit, the algebra of limits can be applied to give the following (and pretty similar looking) theorem:

The Algebra of Continuous Functions If $f(x)$ and $g(x)$are continuous at $x = a$, then so are:

1. $\lambda f + \mu g$,
2. fg,
3. $\frac{f}{g}$ where $g(a) \neq 0$.

There are many theorems that you will learn throughout your degree which will only apply to continuous functions, so it's very important to try to get your head around the notion of continuity!

7.4 Differentiable Functions

Another extremely important application of the notion of a limit is differentiation. You will almost certainly have been differentiating functions for a while now, but you may not have met the formal definition of "differentiable".

Definition $f(x)$ is *differentiable* at $x = a$ if $\lim_{x\to a}\left[\frac{f(x)-g(x)}{x-a}\right]$ exists.

This limit is called the *derivative* of f at a and denoted $\frac{df}{dx}$ or $f'(a)$ (either of these notations is fine to use!).

Example We can use this definition to show that, for example, for $f(x) = x$, $\frac{df}{dx} = 1$ - a result which you will hopefully already be quite comfortable with! So...

$$\begin{aligned}
\frac{df}{dx} &= \lim_{x\to a}\left[\frac{f(x)-g(x)}{x-a}\right] \\
&= \lim_{x\to a}\left[\frac{x-a}{x-a}\right] \\
&= \lim_{x\to a} 1 \\
&= 1
\end{aligned}$$

Before looking at the answer below, try to find $\frac{d}{dx}(x^2)$ using the definition of $\frac{df}{dx}$.

... No cheating!

Answer

$$\begin{aligned}
\frac{d}{dx}(x^2) &= \lim_{x\to a}\left[\frac{x^2-a^2}{x-a}\right] \\
&= \lim_{x\to a}\left[\frac{(x-a)(x+a)}{x-a}\right] \\
&= \lim_{x\to a} x + a \\
&= \lim_{x\to a} x + \lim_{x\to a} a && \text{(by AOL)} \\
&= a + a \\
&= 2a
\end{aligned}$$

We will now use the definition of the derivative to prove the following rules of differentiation, which you should already be familiar with.

The Algebra of Derivatives

If $f(x)$ and $g(x)$ are differentiable at $x = a$ then:

1. $\lambda f + \mu g$ is differentiable at $x = a$ and $(\lambda f + \mu g)'(a) = \lambda f'(a) + \mu g'(a)$.
2. $(fg)'(a) = f'(a)g(a) + g'(a)f(a)$. You may have called this the "*product rule*".
3. For $g \neq 0$, $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$. Commonly known as the "*quotient rule*".

Proof We shall prove parts (2) and (3) first, as they are slightly harder than part (1). Before looking at the proof for part (1), you should have a go at it yourself!

(2) We want to show that $\lim_{x \to a} \left[\frac{f(x)g(x) - f(a)g(a)}{x-a} \right] = f'(a)g(a) + g'(a)f(a)$.

So consider $\frac{f(x)g(x)-f(a)g(a)}{x-a}$...

$$\frac{f(x)g(x) - f(a)g(a)}{x-a} = \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a} \quad \text{(there's that trick again)}$$

$$= f(x) \left[\frac{g(x) - g(a)}{x-a} \right] + g(a) \left[\frac{f(x) - f(a)}{x-a} \right]$$

But we know that

1. $f(x) \to f(a)$ as $x \to a$;
2. $\frac{g(x)-g(a)}{x-a} \to g'(a)$ as $x \to a$, since g is differentiable at $x = a$;
3. $g(x) \to g(a)$ as $x \to a$;
4. $\frac{f(x)-f(a)}{x-a} \to f'(a)$ as $x \to a$, since f is differentiable at $x = a$.

So, using the algebra of limits, we have:

$$f(x) \left[\frac{g(x) - g(a)}{x-a} \right] + g(a) \left[\frac{f(x) - f(a)}{x-a} \right] \to f(a)g'(a) + g(a)f'(a) \qquad \text{as } x \to a$$

(3) We have...

$$\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \frac{f(x)g(a) - f(a)g(x)}{(x-a)g(x)g(a)}$$

$$= \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{(x-a)g(x)g(a)}$$

$$= \frac{[f(x) - f(a)]}{x-a} \frac{g(a)}{g(x)g(a)} + \frac{f(a)}{g(x)g(a)} \frac{[g(a) - g(x)]}{(x-a)}$$

$$= \frac{[f(x) - f(a)]}{x-a} \frac{g(a)}{g(x)g(a)} - \frac{f(a)}{g(x)g(a)} \frac{[g(x) - g(a)]}{(x-a)}$$

But we know that

1. $\frac{f(x)-f(a)}{x-a} \to f'(a)$ as $x \to a$;
2. $\frac{g(a)}{g(x)g(a)} \to \frac{g(a)}{g^2(a)}$ as $x \to a$;

3. $\frac{f(a)}{g(x)g(a)} \to \frac{f(a)}{g^2(a)}$ as $x \to a$;

4. $\frac{g(x)-g(a)}{x-a} \to g'(a)$ as $x \to a$.

So, finally (drumroll):

$$\begin{aligned}
\lim_{x\to a}\left[\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a}\right] &= f'(a)\frac{g(a)}{g^2(a)} - \frac{f(a)}{g^2(a)}g'(a) \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}
\end{aligned}$$

and we've finished!

Now try your hand at proving part (1), we promise it's not as complicated as the proofs above!

... Don't look!

Proof We want $\lim_{x\to a}\left[\frac{\lambda f(x) + \mu g(x) - (\lambda f(a) + \mu g(a))}{x-a}\right] = \lambda f'(a) + \mu g'(a)$. So let's just dive in:

$$\begin{aligned}
\lim_{x\to a}\left[\frac{\lambda f(x) + \mu g(x) - (\lambda f(a) + \mu g(a))}{x-a}\right] &= \lim_{x\to a}\left[\frac{\lambda f(x) + \mu g(x) - \lambda f(a) - \mu g(a)}{x-a}\right] \\
&= \lim_{x\to a}\left[\frac{\lambda(f(x) - f(a))}{x-a} + \frac{\mu(g(x) - g(a))}{x-a}\right] \\
&= \lambda\lim_{x\to a}\left[\frac{f(x) - f(a)}{x-a}\right] + \mu\lim_{x\to a}\left[\frac{g(x) - g(a)}{x-a}\right] \\
&= \lambda f'(a) + \mu g'(a)
\end{aligned}$$