JAN 12

1. Differentiate with respect to x, giving your answer in its simplest form,

(a) $x^2 \ln(3x)$ (4)

(b) $\dfrac{\sin 4x}{x^3}$ (5)

a) $y = uv \quad \dfrac{dy}{dx} = vu' + uv'$ $\quad u = x^2 \quad v = \ln(3x)$

$u' = 2x \quad v' = \dfrac{3}{3x} = \dfrac{1}{x}$

$\therefore \dfrac{dy}{dx} = 2x\ln(3x) + x$

b) $y = \dfrac{u}{v} \quad \dfrac{dy}{dx} = \dfrac{vu' - uv'}{v^2}$ $\quad u = \sin 4x \quad v = x^3$

$u' = 4\cos 4x \quad v' = 3x^2$

$\therefore \dfrac{dy}{dx} = \dfrac{4x^3\cos 4x - 3x^2\sin 4x}{x^6} = \dfrac{4x\cos 4x - 3\sin 4x}{x^4}$

2.

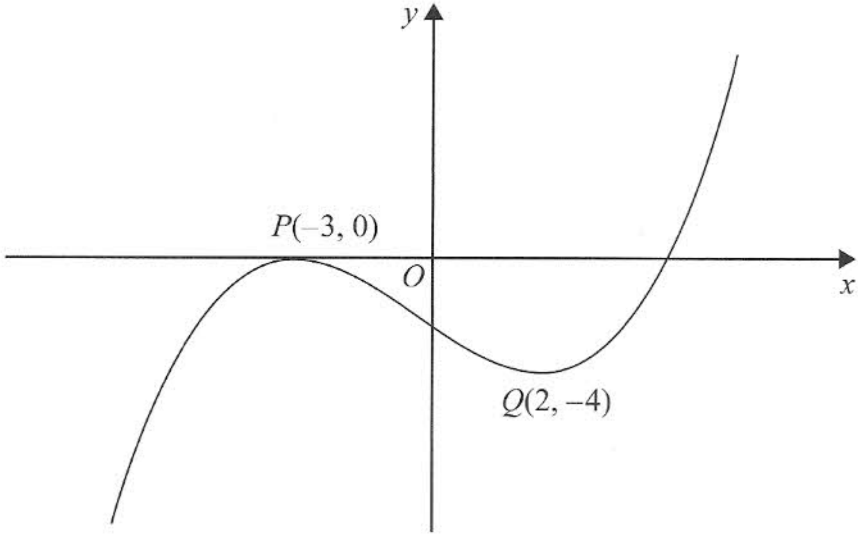


Figure 1

Figure 1 shows the graph of equation $y = \mathrm{f}(x)$.

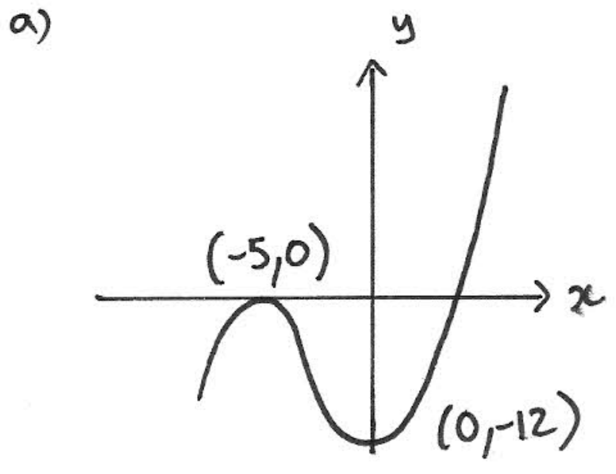
The points $P(-3, 0)$ and $Q(2, -4)$ are stationary points on the graph.

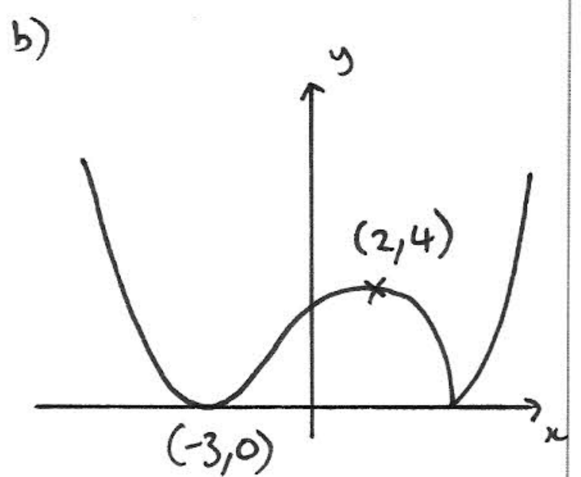
Sketch, on separate diagrams, the graphs of

(a) $y = 3\mathrm{f}(x+2)$ $\updownarrow y\times 3$ $\xleftarrow[2]{}$ $x-2$ **(3)**

(b) $y = |\mathrm{f}(x)|$ bounce **(3)**

On each diagram, show the coordinates of any stationary points.





3. The area, A mm^2, of a bacterial culture growing in milk, t hours after midday, is given by

$$A = 20e^{1.5t}, \quad t \geqslant 0$$

(a) Write down the area of the culture at midday. $t = 0 \Rightarrow \underline{A = 20}$

(1)

(b) Find the time at which the area of the culture is twice its area at midday. Give your answer to the nearest minute.

(5)

b) $2 \times \cancel{20} = \cancel{20}e^{1.5t} \Rightarrow \ln 2 = 1.5t \Rightarrow t = 0.462\ldots$

$\therefore \underline{12{:}28}$

4. The point P is the point on the curve $x = 2\tan\left(y + \frac{\pi}{12}\right)$ with y-coordinate $\frac{\pi}{4}$.

Find an equation of the normal to the curve at P.

(7)

$$x_1 = 2\tan\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = 2\tan\left(\frac{\pi}{3}\right) = 2\sqrt{3}$$

$$\frac{dx}{dy} = 2\sec^2\left(y + \frac{\pi}{12}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{2}\cos^2\left(y + \frac{\pi}{12}\right)$$

$$\text{at } P \quad M_t = \frac{1}{2}\cos^2\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right)\right]^2 = \frac{1}{2} \times \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{8}$$

$$\therefore M_n = -8$$

$$\therefore y - y_1 = m(x - x_1) \Rightarrow y - \frac{\pi}{4} = -8(x - 2\sqrt{3})$$

5. Solve, for $0 \leqslant \theta \leqslant 180°$,

$$2\cot^2 3\theta = 7\,\text{cosec}\,3\theta - 5$$

Give your answers in degrees to 1 decimal place.

(10)

$\frac{\sin^2}{\sin^2} + \frac{\cos^2}{\sin^2} = \frac{1}{\sin^2} \Rightarrow 1 + \cot^2 = \text{cosec}^2$

$\Rightarrow \cot^2 = \text{cosec}^2 - 1$

$\Rightarrow 2\,\text{cosec}^2 3\theta - 2 = 7\,\text{cosec}\,3\theta - 5$

$\Rightarrow 2\,\text{cosec}^2 3\theta - 7\,\text{cosec}\,3\theta + 3 = 0$

$\Rightarrow (2\,\text{cosec}\,3\theta - 1)(\text{cosec}\,3\theta - 3) = 0$

$\text{cosec}\,3\theta = \frac{1}{2}$ $\quad\quad$ $\text{cosec}\,3\theta = 3$

$\sin 3\theta = 2$ $\quad\quad$ $\sin 3\theta = \frac{1}{3}$

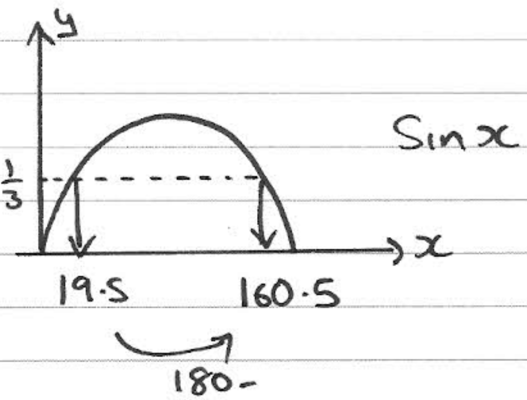
no solutions! $\quad\quad$ $\Rightarrow 3\theta = \sin^{-1}\left(\frac{1}{3}\right)$

180 − $\quad\quad$ +360

$\Rightarrow 3\theta = 19.47\ldots, 160.53\ldots, 379.47\ldots, 520.53$

+360

$\therefore \theta = 6.5°, 53.5°, 126.5°, 173.5°$



6. $$f(x) = x^2 - 3x + 2\cos\left(\tfrac{1}{2}x\right), \quad 0 \leqslant x \leqslant \pi$$

(a) Show that the equation $f(x) = 0$ has a solution in the interval $0.8 < x < 0.9$

(2)

The curve with equation $y = f(x)$ has a minimum point P.

(b) Show that the x-coordinate of P is the solution of the equation

$$x = \frac{3 + \sin\left(\tfrac{1}{2}x\right)}{2}$$

(4)

(c) Using the iteration formula

$$x_{n+1} = \frac{3 + \sin\left(\tfrac{1}{2}x_n\right)}{2}, \quad x_0 = 2$$

find the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

(3)

(d) By choosing a suitable interval, show that the x-coordinate of P is 1.9078 correct to 4 decimal places.

(3)

a) $\left.\begin{array}{l} f(0.8) = 0.082 \\ f(0.9) = -0.089 \end{array}\right\}$ change of sign $\Rightarrow$ root lies between 0.8 and 0.9

b) min point $\Rightarrow f'(x) = 0 \quad \therefore 2x - 3 - \sin\left(\tfrac{1}{2}x\right) = 0$

$\Rightarrow 2x = 3 + \sin\left(\tfrac{1}{2}x\right) \quad \Rightarrow x = \dfrac{3 + \sin\left(\tfrac{1}{2}x\right)}{2}$

c) $x_0 = 2$; $x_1 = 1.921$; $x_2 = 1.910$; $x_3 = 1.908$

d) $f'(1.90775) = -0.00016$
$f'(1.90785) = 0.0000077$

change of sign $\Rightarrow f'(x) = 0$ when x is between 1.90775 and 1.90785

$\therefore x = 1.9078$ (4dp)

7. The function f is defined by

$$f : x \mapsto \frac{3(x+1)}{2x^2+7x-4} - \frac{1}{x+4}, \qquad x \in \mathbb{R}, x > \frac{1}{2}$$

(a) Show that $f(x) = \dfrac{1}{2x-1}$ **(4)**

(b) Find $f^{-1}(x)$ **(3)**

(c) Find the domain of f^{-1} **(1)**

$$g(x) = \ln(x+1)$$

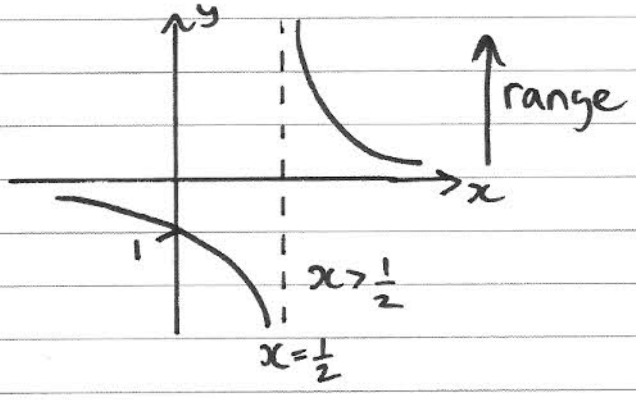
(d) Find the solution of $fg(x) = \dfrac{1}{7}$, giving your answer in terms of e. **(4)**

a) $\dfrac{3(x+1)}{(2x-1)(x+4)} - \dfrac{1 \times (2x-1)}{(x+4) \times (2x-1)} = \dfrac{3x+3-2x+1}{(2x-1)(x+4)}$

$= \dfrac{x+4}{(2x-1)(x+4)} = \dfrac{1}{2x-1}$ #

b) $x = \dfrac{1}{2y-1} \Rightarrow 2y-1 = \dfrac{1}{x} \Rightarrow 2y = \dfrac{1}{x} + 1 \Rightarrow y = f^{-1}(x) = \dfrac{1}{2x} + \dfrac{1}{2}$

c) domain of $f'(x)$ = range of $f(x)$



$y = \dfrac{1}{2x-1}$ $x \neq \dfrac{1}{2}$ asymptote

$x = 0 \Rightarrow y = -1$ $(0, -1)$

$\therefore$ range of $f(x)$ $y > 0$

$\therefore$ domain of $f^{-1}(x)$ $x > 0$

d) $fg(x) = f[\ln(x+1)] = \dfrac{1}{2\ln(x+1)-1} = \dfrac{1}{7}$

$\therefore\ 2\ln(x+1)-1 = 7 \Rightarrow 2\ln(x+1) = 8 \Rightarrow \ln(x+1) = 4$

$\Rightarrow x+1 = e^4 \Rightarrow x = \underline{-1+e^4}$

8. (a) Starting from the formulae for $\sin(A+B)$ and $\cos(A+B)$, prove that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(4)

(b) Deduce that

$$\tan\left(\theta + \frac{\pi}{6}\right) = \frac{1 + \sqrt{3}\tan\theta}{\sqrt{3} - \tan\theta}$$

(3)

(c) Hence, or otherwise, solve, for $0 \leqslant \theta \leqslant \pi$,

$$1 + \sqrt{3}\tan\theta = (\sqrt{3} - \tan\theta)\tan(\pi - \theta)$$

Give your answers as multiples of π.

(6)

a) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$

$= \dfrac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A}\cancel{\cos B}}{\cancel{\cos A}\cancel{\cos B}} - \dfrac{\sin A\sin B}{\cos A\cos B}} \Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ #

b) $\tan\left(\theta + \frac{\pi}{6}\right) = \dfrac{\tan\theta + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\theta\tan\left(\frac{\pi}{6}\right)} = \dfrac{\tan\theta + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}\tan\theta}$ $\times\sqrt{3}$ $\times\sqrt{3}$

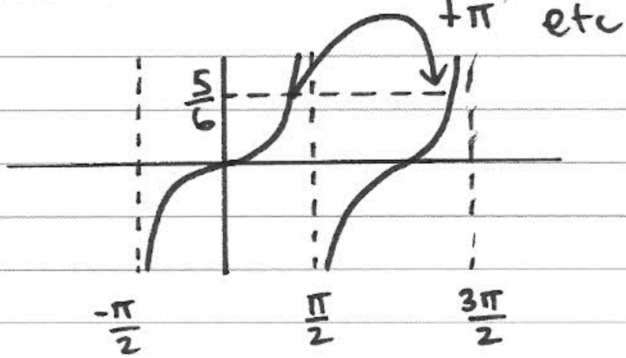
$= \dfrac{\sqrt{3}\tan\theta + 1}{\sqrt{3} - \tan\theta}$ #

c) $1 + \sqrt{3}\tan\theta = (\sqrt{3} - \tan\theta)\tan(\pi - \theta)$

$\Rightarrow \tan\left(\theta + \frac{\pi}{6}\right) = \dfrac{\cancel{(\sqrt{3} - \tan\theta)}\tan(\pi - \theta)}{\cancel{\sqrt{3} - \tan\theta}}$

$\therefore \tan\left(\theta + \frac{\pi}{6}\right) = \tan(\pi - \theta) \quad \Rightarrow \theta + \frac{\pi}{6} = \pi - \theta$

$\Rightarrow 2\theta = \frac{5}{6}\pi$



$$2\theta = \frac{5}{6}\pi, \frac{11\pi}{6}, \frac{17\pi}{6}, \dots$$

$$\theta = \frac{5}{12}\pi, \frac{11\pi}{12}$$

alt

$\tan\pi = 0$

$$\tan(\pi-\theta) = \frac{\tan\pi - \tan\theta}{1+\tan(\pi)\tan\theta} = \frac{-\tan\theta}{1} = -\tan\theta$$

$$\Rightarrow 1+\sqrt{3}\tan\theta = (\sqrt{3}-\tan\theta)\times -\tan\theta$$

$$\Rightarrow 1+\sqrt{3}\tan\theta = -\sqrt{3}\tan\theta + \tan^2\theta$$

$$\Rightarrow \tan^2\theta - 2\sqrt{3}\tan\theta - 1 = 0$$

$$\Rightarrow \tan\theta = \frac{2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4(-1)}}{2} = \frac{2\sqrt{3} \pm 4}{2} = \sqrt{3} \pm 2$$

$$\theta = \tan^{-1}(\sqrt{3}+2) \Rightarrow \theta = \frac{5\pi}{12}$$

$$\theta = \tan^{-1}(\sqrt{3}-2) \Rightarrow \theta = -\frac{1}{12}\pi \xrightarrow{+\pi} \frac{11}{12}\pi$$

$$\therefore \theta = \frac{5\pi}{12}, \frac{11\pi}{12}$$