QUESTION 3

$$x^3 - 3px + q \equiv a(x-\alpha)^3 + b(x-\beta)^3$$

$$\Rightarrow x^3 - 3px + q \equiv a\left(x^3 - 3\alpha x^2 + 3\alpha^2 x - \alpha^3\right) + b\left(x^3 - 3\beta x^2 + 3\beta^2 x - \beta^3\right)$$

$$\Rightarrow \begin{cases} a+b=1 & (1) \\ a\alpha + b\beta = 0 & (2) \\ a\alpha^2 + b\beta^2 = -p & (3) \\ a\alpha^3 + b\beta^3 = -q & (4) \end{cases}$$

Now (1) multiplied by α and subtracted from (2) gives :-

$$b(\alpha-\beta) = \alpha \Rightarrow b = \frac{\alpha}{\alpha-\beta}, a = \frac{-\beta}{\alpha-\beta} \quad \alpha \neq \beta$$

Substituting these values into (3) gives :-

$$-\frac{\beta\alpha^2}{\alpha-\beta} + \frac{\alpha\beta^2}{\alpha-\beta} = -p \Rightarrow \frac{\alpha\beta(\beta-\alpha)}{\alpha-\beta} = -p \Rightarrow \alpha\beta = p \quad \alpha \neq \beta$$

Again, substituting the values into (4) gives :-

$$-\frac{\beta\alpha^3}{\alpha-\beta} + \frac{\alpha\beta^3}{\alpha-\beta} = -q \Rightarrow \frac{\alpha\beta\left(\beta^2-\alpha^2\right)}{\alpha-\beta} = -q \Rightarrow \frac{\alpha\beta(\beta+\alpha)(\beta-\alpha)}{\alpha-\beta} = -q$$

$$\Rightarrow \alpha\beta(\alpha+\beta) = q \Rightarrow \alpha+\beta = \frac{q}{p} \quad \alpha \neq \beta$$

Therefore α, β are the roots of the quadratic equation :-

$$t^2 - \frac{q}{p}t + p = 0 \Rightarrow pt^2 - qt + p^2 = 0$$

Further the condition that $\alpha \neq \beta$ is contained in the idea that the discriminant of this equation is not zero i.e :-

$$(-q)^2 - 4 \times p \times p^2 \neq 0 \Rightarrow q^2 \neq 4p^3$$

The conditions that $p \neq 0, q \neq 0$ ensure that the equation is non-trivial.

Now $x^3 - 24x + 48 = 0 \Rightarrow p = 8, q = 48$ and $4p^3 = 2048 \neq 2304 = q^2$

Therefore our quadratic is $8t^2 - 48t + 64 = 0$

$$\Rightarrow t^2 - 6t + 8 = 0 \Rightarrow (t-2)(t-4) = 0 \Rightarrow t = 2 \text{ or } 4$$

Hence $\alpha = 2, \beta = 4 \Rightarrow a = 2, b = -1$ so that :-

$$x^3 - 24x + 48 = 2(x-2)^3 - (x-4)^3 = 0$$

$$\Rightarrow 2 = \left(\frac{x-4}{x-2}\right)^3 \Rightarrow 2^{\frac{1}{3}} = \frac{x-4}{x-2} = 1 - \frac{2}{x-2}$$

$$\Rightarrow x - 2 = \frac{2}{1-2^{\frac{1}{3}}} \Rightarrow x = 2 + \frac{2}{1-2^{\frac{1}{3}}} = \frac{2\left(2-2^{\frac{1}{3}}\right)}{1-2^{\frac{1}{3}}}$$

There are actual three cube roots of 2; what we have dealt with so far is the one that is real. The other two are:-

$$2^{\frac{1}{3}} e^{\frac{2\pi i}{3}} \left(= 2^{\frac{1}{3}} \omega \right) \text{ and } 2^{\frac{1}{3}} e^{\frac{4\pi i}{3}} \left(= 2^{\frac{1}{3}} \omega^2 \right)$$

Now $2^{\frac{1}{3}} w = \dfrac{x-4}{x-2} \Rightarrow x = \dfrac{2\left(2-2^{\frac{1}{3}}\omega\right)}{1-2^{\frac{1}{3}}\omega}$ and $2^{\frac{1}{3}} w^2 = \dfrac{x-4}{x-2} \Rightarrow x = \dfrac{2\left(2-2^{\frac{1}{3}}\omega^2\right)}{1-2^{\frac{1}{3}}\omega^2}$

Finally $x^3 - 3r^2x + 2r^3 = 0 \Rightarrow (x-r)\left(x^2 + rx - 2r^2\right) = 0$

$$\Rightarrow (x-r)(x-r)(x+2r) = 0$$

$$\Rightarrow x = r, r, -2r$$

This is the case $q^2 = 4p^3$ and demonstrates a repeated real root.
