QUESTION 8

$$\omega=\frac{1+iz}{i+z} \text{ where } z=x+iy \Rightarrow \omega=\frac{1+i(x+iy)}{i+(x+iy)}=\frac{(1-y)+ix}{x+i(1+y)}$$

$$=\frac{(1-y)+ix}{x+i(1+y)}\times\frac{x-i(1+y)}{x-i(1+y)}$$

$$=\frac{2x+i\left(x^2+y^2-1\right)}{x^2+(1+y)^2}$$

Hence $u=\dfrac{2x}{x^2+(1+y)^2}, v=\dfrac{x^2+y^2-1}{x^2+(1+y)^2}$

Part (i)

Now $y=0 \Rightarrow u=\dfrac{2x}{x^2+1^2}, v=\dfrac{x^2-1}{x^2+1}$

Therefore $u^2+v^2=\left(\dfrac{2x}{x^2+1^2}\right)^2+\left(\dfrac{x^2-1}{x^2+1}\right)^2=\dfrac{4x^2+x^4-2x^2+1}{\left(x^2+1\right)^2}=\dfrac{\left(x^2+1\right)^2}{\left(x^2+1\right)^2}=1$

However $-1\le\dfrac{x^2-1}{x^2+1}<1$ for $-\infty<x<\infty$ so that $-1\le v<1$.

Therefore the locus of ω is the circle $u^2+v^2=1$ without the point $(0,1)$.

Part (ii)

Now $-1<x<1 \Rightarrow -1<\dfrac{2x}{x^2+1}<1 \Rightarrow -1<u<1$ with $y=0$.

Further $-1<x<1 \Rightarrow -1\le\dfrac{x^2-1}{x^2+1}\le 0 \Rightarrow -1\le v\le 0$ with $y=0$.

Therefore the locus of ω is the semi-circle $u^2+v^2=1$ below the x axis i.e without the points $(1,0),(-1,0)$.

Part (iii)

Now $x=0,-1<y<1 \Rightarrow u=0,-\infty<\frac{y^2-1}{(1+y)^2}<0 \Rightarrow -\infty<v<0$

Therefore the locus of ω is all the imaginary axis that has negative values.

Part (iv)

$$y=1 \Rightarrow u=\frac{2x}{x^2+4}, v=\frac{x^2}{x^2+4}$$

$$\frac{du}{dx}=\frac{(x^2+4)\times 2-2x\times 2x}{(x^2+4)^2}=\frac{2(4-x^2)}{(x^2+4)^2} \Rightarrow \frac{du}{dx}=0 \text{ when } x=\pm 2$$

Therefore $-\frac{1}{2}\le u\le\frac{1}{2}$ as $-\infty<x<\infty$.

Clearly $0\le v<1$ for $-\infty<x<\infty$

Further $u^2=v(1-v) \Rightarrow u^2+\left(v-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$

Since we have excluded $v=1$ then we omit the point $(0,1)$.

Therefore the locus of ω is the circle $u^2+\left(v-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$ omitting the point $(0,1)$.
