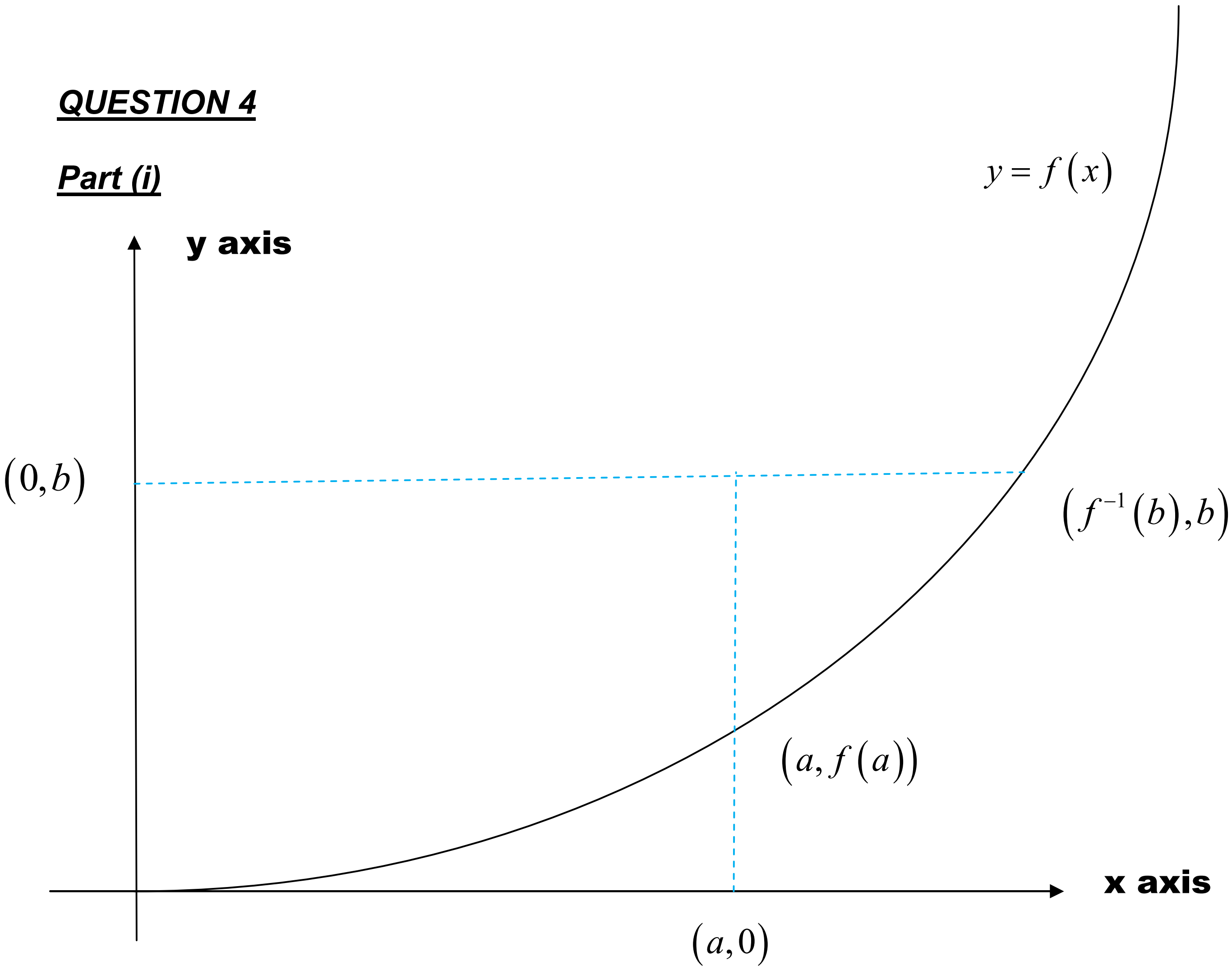
QUESTION 4

Part (i)



The area between the curve, the y axis and the line $y=b$ is $\int_0^b f^{-1}(y)\,dy$.

The area between the curve, the x axis and the line $x=a$ is $\int_0^a f(x)\,dx$.

The area enclosed by the axes and the blue lines is ab.

Clearly $ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$

Equality holds when $a = f^{-1}(b) \Rightarrow f(a) = b$.

Part (ii)

$f(x) = x^{p-1}$ with $p>1 \Rightarrow f^{-1}(x) = x^{\frac{1}{p-1}}$

Now $\frac{1}{p}+\frac{1}{q}=1$ with $p\neq 1 \Rightarrow p=\frac{1}{1-\frac{1}{q}}=\frac{q}{q-1}$

$$\Rightarrow p-1=\frac{q}{q-1}-1=\frac{1}{q-1}$$

$$\Rightarrow \frac{1}{p-1}=q-1$$

Hence $f^{-1}\left(x\right)=x^{q-1}$

Now $\int_0^a f\left(x\right)dx=\int_0^a x^{p-1}dx=\left[\frac{x^p}{p}\right]_0^a=\frac{a^p}{p}$

Further $\int_0^b f^{-1}\left(x\right)dx=\int_0^b x^{q-1}dx=\left[\frac{x^q}{q}\right]_0^b=\frac{b^q}{q}$

Therefore using our formula $ab\leq\frac{a^p}{p}+\frac{b^q}{q}$

Now $\frac{1}{p}+\frac{1}{q}=1\Rightarrow 1+\frac{p}{q}=p\Rightarrow\frac{p}{q}=p-1$

So that $f\left(a\right)=b\Rightarrow a^{p-1}=b\Rightarrow a^{\frac{p}{q}}=b\Rightarrow a^p=b^q$

Therefore $\frac{a^p}{p}+\frac{b^q}{q}=a^p\left(\frac{1}{p}+\frac{1}{q}\right)=a^p=a\times a^{p-1}=ab$

<u>Part (iii)</u>

Using $f\left(x\right)=\sin x$ we have:- $\int_0^a f\left(x\right)dx=\int_0^a \sin x dx=\left[-\cos x\right]_0^a=1-\cos a$

Now $\int_0^b f^{-1}\left(x\right)dx=\int_0^b \sin^{-1}x dx=\left[x\sin^{-1}x\right]_0^b-\int_0^b\frac{x}{\sqrt{1-x^2}}dx$

$$=b\sin^{-1}b+\left[\sqrt{1-x^2}\right]_0^b$$

$$=b\sin^{-1}b+\sqrt{1-b^2}-1$$

Therefore using our formula $ab \le \left(b\sin^{-1} b + \sqrt{1-b^2} - 1\right) + \left(1 - \cos a\right)$

$$\Rightarrow ab \le b\sin^{-1} b + \sqrt{1-b^2} - \cos a$$

Choosing $b = \frac{1}{t}$ with $t \ge 1$ and $a = 0$ we have:-

$$0 \le \frac{1}{t}\sin^{-1}\frac{1}{t} + \sqrt{1-\left(\frac{1}{t}\right)^2} - 1 \Rightarrow 0 \le \sin^{-1}\frac{1}{t} + \sqrt{t^2-1} - t$$

$$\Rightarrow t - \sqrt{t^2-1} \le \sin^{-1}\frac{1}{t}$$
