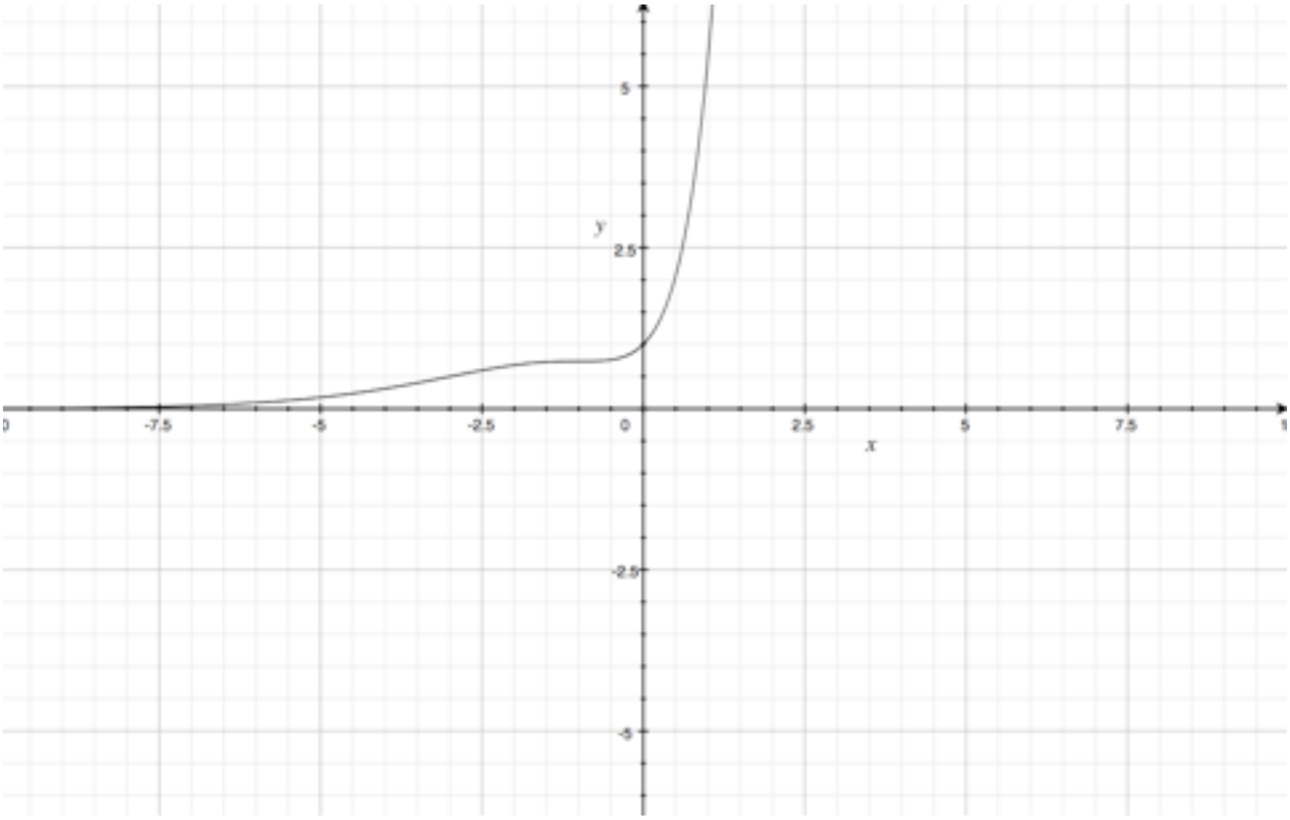
STEP III 1999 Q2

i) We have $f'(x) = (1 + x^2)e^x + 2xe^x$ by the product rule, which simplifies to $f'(x) = (1 + x)^2 e^x$
$(1 + x)^2 \geq 0$ as it is square and $e^x > 0$ so their product is ≥ 0, as required.

Sketch:



There is an inflection point at x=-1, where f'(x)=0 and the y-intercept is 1.

The second part is obvious from the graph, lines y=k will intersect the graph once for k>0 and not at all for k≤0, as the graph has an asymptote y=0.

ii) Let $g(x) = e^x - 1 - k\tan^{-1}(x)$

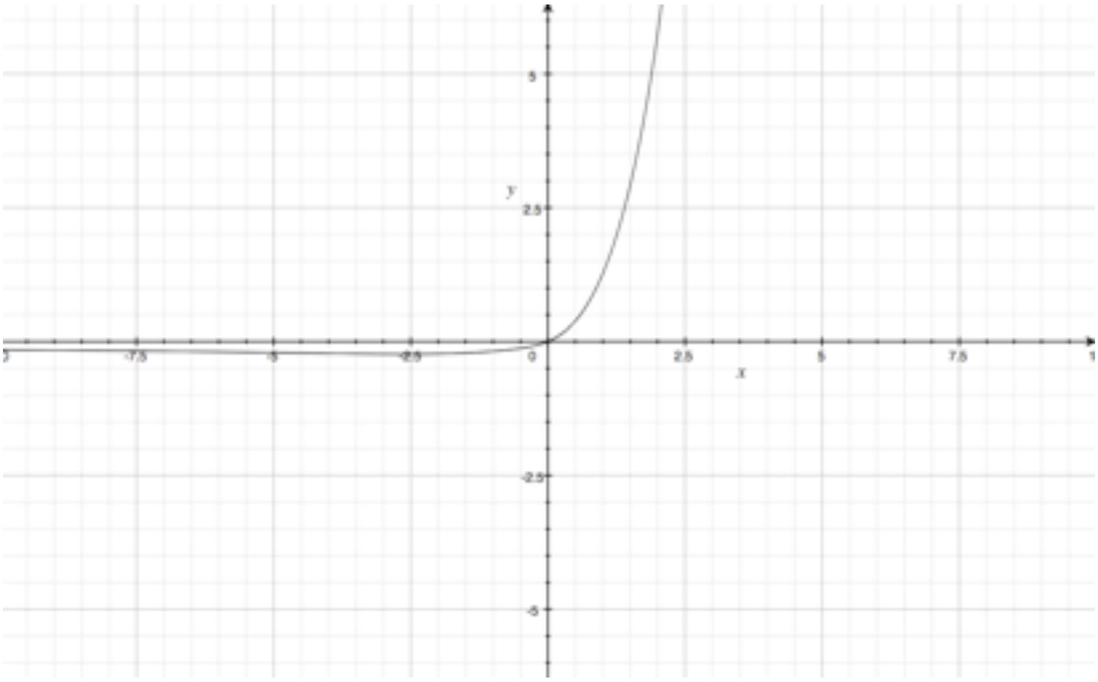
Then $g'(x) = e^x - \dfrac{k}{1 + x^2}$ so for turning points we require $0 = e^x - \dfrac{k}{1 + x^2} \Rightarrow (1 + x^2)e^x = k$

which has exactly 1 solution for k>0 (from the first part of the question). So g(x) has exactly one turning point.
We observe that x=0 is a root for all values of k. Therefore the turning point must occur for g(x)≤0. We observe that k<1 means that x<0 (this can be seen from the sketch in part i). So the turning point occurs for -∞<x<0. It is obvious that the turning point must be a minimum (consider what happens for large positive x, there is no maximum of g(x)).
We consider what happens to g(x) for large negative x.

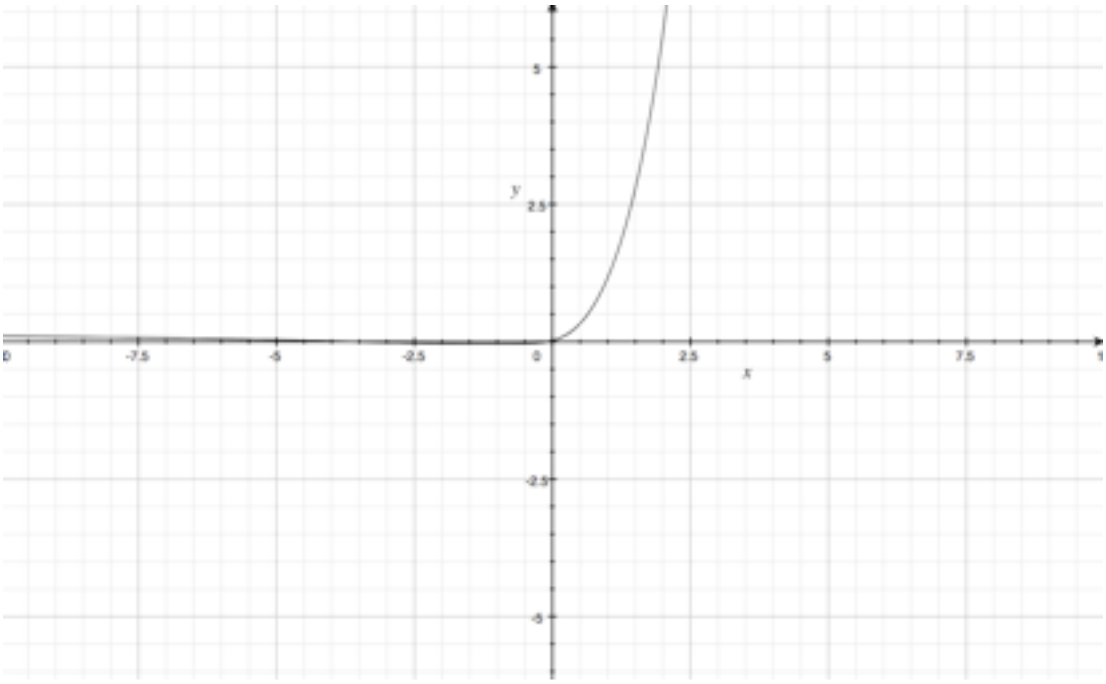
$$\lim_{x \to -\infty} e^x - 1 - k\tan^{-1}(x) = -1 + k\frac{\pi}{2}$$

Therefore there is an asymptote at $g(x) = A = -1 + k\dfrac{\pi}{2}$

When this asymptote is below the x-axis (or the x=axis itself), the only root is at x=0
Sketch:



When this asymptote is above the x-axis, there must be another root as the function must pass below the x-axis to pass through (0,0).
Sketch:



So for $0 < k \le \frac{2}{\pi}$, $A = -1 + k\frac{\pi}{2} \le 0$ so there is 1 real root. For $\frac{2}{\pi} < k < 1$, $A = -1 + k\frac{\pi}{2} > 0$ so there are 2 real roots.