Question 6

$$|z-k|^2 = r^2 \Rightarrow (z-k)(z-k)^* = r^2 \Rightarrow (z-k)(z^* - k^*) = r^2$$

$$\Rightarrow zz^* - kz^* - k^*z + kk^* - r^2 = 0$$

$$k = i, r = 1 \Rightarrow zz^* - iz^* + iz = 0$$

$$w_1 = \frac{z}{z-1} \Rightarrow z = \frac{w_1}{w_1 - 1} \Rightarrow z^* = \frac{w_1^*}{w_1^* - 1}$$

Substituting:- $\left(\frac{w_1}{w_1-1}\right) \times \left(\frac{w_1^*}{w_1^*-1}\right) - i \times \left(\frac{w_1^*}{w_1^*-1}\right) + i \times \left(\frac{w_1}{w_1-1}\right) = 0$

$$\Rightarrow w_1 w_1^* - iw_1^*(w_1 - 1) + iw_1(w_1^* - 1) = 0$$

$$\Rightarrow w_1 w_1^* + iw_1^* - iw_1 = 0$$

$$\Rightarrow (w_1 + i)(w_1^* - i) = 0$$

$$\Rightarrow (w_1 + i)(w_1 + i)^* = 0$$

This is clearly a circle centre $-i$ and radius 1.

$$w_2 = z^* \Rightarrow z = w_2^*$$

Substituting:- $w_2^* w_2 - iw_2 + iw_2^* = 0 \Rightarrow (w_2 + i)(w_2 + i)^* = 0$ as before.

This is clearly the same circle as before.

For $w_1 = \dfrac{z}{z-1}$ and $w_2 = z^*$ to coincide we must have:-

$$w_1 = w_2 \Rightarrow \frac{z}{z-1} = z^* \Rightarrow z + z^* = zz^* \Rightarrow zz^* - z^* - z = 0 \Rightarrow (z-1)(z-1)^* = 1$$

Which is a circle of radius 1 and centre $(1,0)$.

In order for z to satisfy both its original equation and this new one (in other words to make w_1 and w_2 coincide) we need the intersection points of the original circle and this new one which are the origin and the point $1+i$.