K

WELSH JOINT EDUCATION COMMITTEE
General Certificate of Education
Advanced Level/Special Paper

CYD-BWYLLGOR ADDYSG CYMRU
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PURE MATHEMATICS S

A.M. FRIDAY, 2 July 1999

(3 Hours)

INSTRUCTIONS TO CANDIDATES

Answer **six** questions.

INFORMATION FOR CANDIDATES

An electronic calculator will be required.

The booklet "Information for the use of candidates in Mathematics" is available and may be used.

The number of marks is given in brackets at the end of each question or part-question.

You are reminded of the necessity for good English and orderly presentation in your answers.

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1. (*a*) Show that, for $x \neq 0$,

$$x^4 - 8x^3 + 16x^2 - 8x + 1 \equiv x^2\left[\left(x + \frac{1}{x}\right)^2 - 8\left(x + \frac{1}{x}\right) + 14\right].$$

Hence find all the roots of the equation

$$x^4 - 8x^3 + 16x^2 - 8x + 1 = 0$$

giving your answers correct to three decimal places. [8]

(*b*) Find **all** the real values of k for which the equation

$$x^4 - 8x^3 + kx^2 - 8x + 1 = 0$$

has at least one repeated root. [9]

2. (*a*) The curve C_1 has parametric equations

$$x = \frac{3}{2}\cos\theta;\ y = \frac{\sqrt{5}}{2}\sin\theta.$$

(i) Show that the cartesian equation of C_1 is

$$20x^2 + 36y^2 = 45.$$

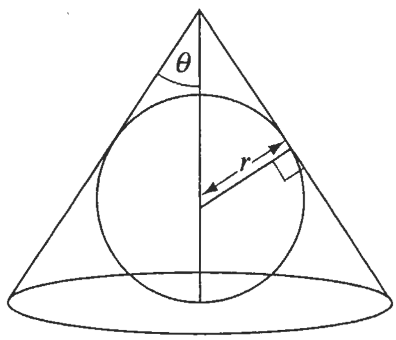
(ii) Given that P is any point on C_1, show that the sum of the distances of P from the points (1, 0) and (–1, 0) is equal to 3. [5]

(*b*) The point Q moves in such a way that the difference between its distances from (1, 0) and (–1, 0) is equal to 1. Show that the cartesian equation of C_2, the path of Q, is

$$12x^2 - 4y^2 = 3.$$ [5]

(*c*) Show that the curves C_1 and C_2 intersect at right-angles at each of their points of intersection. [7]

3.



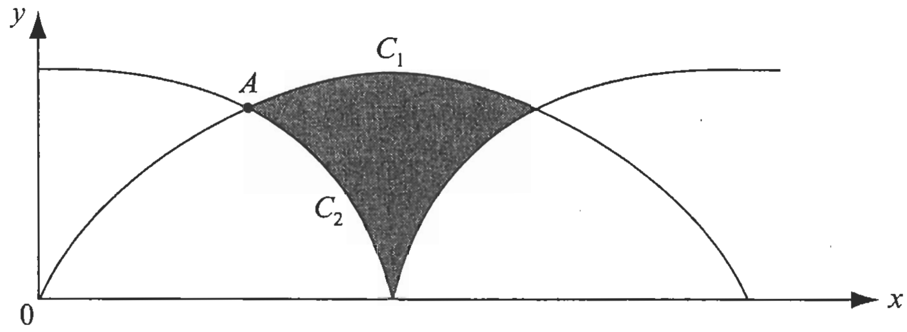
The figure shows a right circular cone having semi-vertical angle θ whose curved surface and base touch a sphere of fixed radius r.

Show that the volume V of the cone is given by

$$V = \frac{1}{3}\pi r^3 (1 + \operatorname{cosec}\theta)^3 \tan^2\theta.$$

Find the minimum value of V as θ varies. [17]

4.



The diagram shows the two curves C_1 and C_2 with parametric equations:

$$C_1\text{: } x = \theta - \sin\theta,\quad y = 1 - \cos\theta \qquad (0 \leqslant \theta \leqslant 2\pi)$$
$$C_2\text{: } x = \phi + \sin\phi,\quad y = 1 + \cos\phi \qquad (0 \leqslant \phi \leqslant 2\pi)$$

(*a*) Show that at A, the point of intersection of C_1 and C_2 marked on the diagram,

$$\phi + \sin\phi = \frac{\pi}{2}.$$

Hence show that $\phi = 0{\cdot}8317$ correct to four decimal places. [6]

(*b*) Show that the area of the region enclosed between C_1 and C_2, the shaded region in the diagram, is approximately 2·5. [11]

5. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ -4 & 0 & 1 \end{bmatrix}.$$

(*a*) Show that

$$\mathbf{A}^3 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$$

where a, b and c are constants to be found. [6]

(*b*) Deduce an expression for $\mathbf{A}^{-1}$ as a quadratic in $\mathbf{A}$ and hence evaluate $\mathbf{A}^{-1}$. [4]

(*c*) Express each of $\mathbf{A}^4$ and $\mathbf{A}^6$ as a quadratic in $\mathbf{A}$.

(i) Conjecture, in terms of n, a general expression for $\mathbf{A}^{2n}$ as a quadratic in $\mathbf{A}$, where n is a positive integer.

(ii) Prove your conjecture using mathematical induction. [7]

6. (*a*) A complex number z satisfies the equation

$$|z - z_1| = \lambda |z - z_2|$$

where z_1, z_2 are fixed unequal complex numbers and λ is a positive constant. The point P represents z in an Argand diagram. Show that the locus of P is a straight line or a circle depending upon the value of λ. [6]

(*b*) The complex numbers z, w are represented by the points P, Q in Argand diagrams and

$$w = \frac{az + b}{cz + d}$$

where a, b, c, d $(ad \neq bc)$ are real constants.

(i) Express z in terms of w.

(ii) Given that $|z - \alpha| = \beta$, where α is a fixed complex number and β is a positive constant, describe the locus of P and show that the locus of Q is a straight line or a circle. State the condition on the constants for the locus of Q to be a straight line.

(iii) Given that the locus of P has equation

$$x^2 + y^2 - 4x - 6y + 3 = 0$$

and that

$$w = \frac{z + 1}{2z - 6}$$

determine whether the locus of Q is a straight line or a circle. [11]

7. (*a*) The function f is defined by

$$f(x) = x^3 - 3ax - b \quad \text{where } a > 0.$$

Show that

(i) f has two turning points,

(ii) the product of the values of $f(x)$ at these two turning points is $b^2 - 4a^3$.

Deduce that the equation

$$f(x) = 0$$

has three distinct real roots only if $b^2 < 4a^3$. [6]

(*b*) The curve C has equation $y^2 = 4ax$.

(i) Obtain the equation of the normal to C at the point $(at^2, 2at)$ and write down the condition for this normal to pass through the point $P(h, k)$.

(ii) Show that there are three distinct normals to C passing through P only if P lies in the region R defined by

$$27ay^2 < 4(x - 2a)^3.$$

(iii) Given that P lies in R, and that $T_1\,(at_1^2, 2at_1)$, $T_2\,(at_2^2, 2at_2)$ and $T_3\,(at_3^2, 2at_3)$ are the three points on C at which the normals to C pass through P, show that

$$t_1 + t_2 + t_3 = 0.$$

(iv) Write down the equation in t whose roots give the parameters of the points of intersection of C and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Deduce that the circle that passes through T_1, T_2, T_3 also passes through the origin. [11]

8. (*a*) Given that

$$I = \int_0^\infty \frac{dx}{a + \cosh x}$$

use the substitution $t = \tanh \frac{1}{2}x$

(i) to find an expression for I when $0 < a < 1$,

(ii) to show that, when $a > 1$,

$$I = \frac{1}{\sqrt{a^2 - 1}} \ln \left[\frac{\sqrt{a+1} + \sqrt{a-1}}{\sqrt{a+1} - \sqrt{a-1}} \right].$$ [11]

(*b*) By considering

$$\frac{d}{dx}\left[\frac{\sinh x}{a + \cosh x} \right]$$

find, when $a > 1$, an expression in terms of a for

$$\int_0^\infty \frac{dx}{(a + \cosh x)^2}.$$ [6]