CHAPTER 1

Review of Algebra

Much of the material in this chapter is revision from GCSE maths (although some of the exercises are harder). Some of it – particularly the work on logarithms – may be new if you have not done A-level maths. If you have done A-level, and are confident, you can skip most of the exercises and just do the worksheet, using the chapter for reference where necessary.

1. Algebraic Expressions

1.1. Evaluating Algebraic Expressions

EXAMPLES 1.1:

(i) A firm that manufactures widgets has m machines and employs n workers. The number of widgets it produces each day is given by the expression $m^2(n-3)$. How many widgets does it produce when $m = 5$ and $n = 6$?

$$\text{Number of widgets} = 5^2 \times (6-3) = 25 \times 3 = 75$$

(ii) In another firm, the cost of producing x widgets is given by $3x^2 + 5x + 4$. What is the cost of producing (a) 10 widgets (b) 1 widget?

$$\begin{aligned} \text{When } x = 10, \text{ cost} &= (3 \times 10^2) + (5 \times 10) + 4 = 300 + 50 + 4 = 354 \\ \text{When } x = 1, \text{ cost} &= 3 \times 1^2 + 5 \times 1 + 4 = 3 + 5 + 4 = 12 \end{aligned}$$

It might be clearer to use brackets here, but they are not essential: the rule is that $\times$ and $\div$ are evaluated before $+$ and $-$.

(iii) Evaluate the expression $8y^4 - \frac{12}{6-y}$ when $y = -2$.
(Remember that y^4 means $y \times y \times y \times y$.)

$$8y^4 - \frac{12}{6-y} = 8 \times (-2)^4 - \frac{12}{6-(-2)} = 8 \times 16 - \frac{12}{8} = 128 - 1.5 = 126.5$$

(If you are uncertain about using negative numbers, work through *Jacques* pp.7–9.)

EXERCISES 1.1: Evaluate the following expressions when $x = 1$, $y = 3$, $z = -2$ and $t = 0$:
(a) $3y^2 - z$ (b) $xt + z^3$ (c) $(x + 3z)y$ (d) $\frac{y}{z} + \frac{2}{x}$ (e) $(x + y)^3$ (f) $5 - \frac{x+3}{2t-z}$

1.2. Manipulating and Simplifying Algebraic Expressions

EXAMPLES 1.2:

(i) Simplify $1 + 3x - 4y + 3xy + 5y^2 + y - y^2 + 4xy - 8$.
This is done by *collecting like terms*, and adding them together:

$$\begin{aligned} & 1 + 3x - 4y + 3xy + 5y^2 + y - y^2 + 4xy - 8 \\ = \;& 5y^2 - y^2 + 3xy + 4xy + 3x - 4y + y + 1 - 8 \\ = \;& 4y^2 + 7xy + 3x - 3y - 7 \end{aligned}$$

The order of the terms in the answer doesn't matter, but we often put a positive term first, and/or write "higher-order" terms such as y^2 before "lower-order" ones such as y or a number.

(ii) Simplify $5(x - 3) - 2x(x + y - 1)$.
Here we need to *multiply out the brackets* first, and then collect terms:

$$\begin{aligned} 5(x-3) - 2x(x+y-1 &= 5x - 15 - 2x^2 - 2xy + 2x \\ &= 7x - 2x^2 - 2xy + 5 \end{aligned}$$

(iii) Multiply x^3 by x^2.

$$x^3 \times x^2 = x \times x \times x \times x \times x = x^5$$

(iv) Divide x^3 by x^2.
We can write this as a fraction, and cancel:

$$x^3 \div x^2 = \frac{x \times x \times x}{x \times x} = \frac{x}{1} = x$$

(v) Multiply $5x^2y^4$ by $4yx^6$.

$$\begin{aligned} 5x^2y^4 \times 4yx^6 &= 5 \times x^2 \times y^4 \times 4 \times y \times x^6 \\ &= 20 \times x^8 \times y^5 \\ &= 20x^8y^5 \end{aligned}$$

Note that you can always change the order of multiplication.

(vi) Divide $6x^2y^3$ by $2yx^5$.

$$\begin{aligned} 6x^2y^3 \div 2yx^5 &= \frac{6x^2y^3}{2yx^5} = \frac{3x^2y^3}{yx^5} = \frac{3y^3}{yx^3} \\ &= \frac{3y^2}{x^3} \end{aligned}$$

(vii) Add $\frac{3x}{y}$ and $\frac{y}{2}$.
The rules for algebraic fractions are just the same as for numbers, so here we find a *common denominator*:

$$\begin{aligned} \frac{3x}{y} + \frac{y}{2} &= \frac{6x}{2y} + \frac{y^2}{2y} \\ &= \frac{6x + y^2}{2y} \end{aligned}$$

(viii) Divide $\frac{3x^2}{y}$ by $\frac{xy^3}{2}$.

$$\begin{aligned}\frac{3x^2}{y} \div \frac{xy^3}{2} &= \frac{3x^2}{y} \times \frac{2}{xy^3} = \frac{3x^2 \times 2}{y \times xy^3} = \frac{6x^2}{xy^4} \\ &= \frac{6x}{y^4}\end{aligned}$$

Exercises 1.2: Simplify the following as much as possible:

(1) (a) $3x - 17 + x^3 + 10x - 8$ (b) $2(x + 3y) - 2(x + 7y - x^2)$

(2) (a) $z^2x - (z + 1) + z(2xz + 3)$ (b) $(x + 2)(x + 4) + (3 - x)(x + 2)$

(3) (a) $\dfrac{3x^2y}{6x}$ (b) $\dfrac{12xy^3}{2x^2y^2}$

(4) (a) $2x^2 \div 8xy$ (b) $4xy \times 5x^2y^3$

(5) (a) $\dfrac{2x}{y} \times \dfrac{y^2}{2x}$ (b) $\dfrac{2x}{y} \div \dfrac{y^2}{2x}$

(6) (a) $\dfrac{2x + 1}{4} + \dfrac{x}{3}$ (b) $\dfrac{1}{x - 1} - \dfrac{1}{x + 1}$ (giving the answers as a single fraction)

1.3. Factorising

A number can be written as the product of its factors. For example: $30 = 5 \times 6 = 5 \times 3 \times 2$. Similarly "factorise" an algebraic expression means "write the expression as the product of two (or more) expressions." Of course, some numbers (primes) don't have any proper factors, and similarly, some algebraic expressions can't be factorised.

Examples 1.3:

(i) Factorise $6x^2 + 15x$.
Here, $3x$ is a common factor of each term in the expression so:

$$6x^2 + 15x = 3x(2x + 5)$$

The factors are $3x$ and $(2x + 5)$. You can check the answer by multiplying out the brackets.

(ii) Factorise $x^2 + 2xy + 3x + 6y$.
There is no common factor of all the terms but the first pair have a common factor, and so do the second pair, and this leads us to the factors of the whole expression:

$$\begin{aligned}x^2 + 2xy + 3x + 6y &= x(x + 2y) + 3(x + 2y) \\ &= (x + 3)(x + 2y)\end{aligned}$$

Again, check by multiplying out the brackets.

(iii) Factorise $x^2 + 2xy + 3x + 3y$.
We can try the method of the previous example, but it doesn't work. The expression can't be factorised.

(iv) Simplify $5(x^2+6x+3)-3(x^2+4x+5)$.
Here we can first multiply out the brackets, then collect like terms, then factorise:

$$\begin{aligned} 5(x^2+6x+3)-3(x^2+4x+5) &= 5x^2+30x+15-3x^2-12x-15 \\ &= 2x^2+18x \\ &= 2x(x+9) \end{aligned}$$

EXERCISES 1.3: **Factorising**

(1) Factorise: (a) $3x+6xy$ (b) $2y^2+7y$ (c) $6a+3b+9c$

(2) Simplify and factorise: (a) $x(x^2+8)+2x^2(x-5)-8x$ (b) $a(b+c)-b(a+c)$

(3) Factorise: $xy+2y+2xz+4z$

(4) Simplify and factorise: $3x(x+\frac{4}{x})-4(x^2+3)+2x$

1.4. Polynomials

Expressions such as

$$5x^2-9x^4-20x+7 \text{ and } 2y^5+y^3-100y^2+1$$

are called *polynomials*. A polynomial in x is a sum of terms, and each term is either a power of x (multiplied by a number called a *coefficient*), or just a number known as a *constant*. All the powers must be positive integers. (Remember: an *integer* is a positive or negative whole number.) The *degree* of the polynomial is the highest power. A polynomial of degree 2 is called a *quadratic* polynomial.

EXAMPLES 1.4: *Polynomials*

(i) $5x^2-9x^4-20x+7$ is a polynomial of degree 4. In this polynomial, the coefficient of x^2 is 5 and the coefficient of x is -20. The constant term is 7.

(ii) x^2+5x+6 is a quadratic polynomial. Here the coefficient of x^2 is 1.

1.5. Factorising Quadratics

In section 1.3 we factorised a quadratic polynomial by finding a common factor of each term: $6x^2+15x=3x(2x+5)$. But this only works because there is no constant term. Otherwise, we can try a different method:

EXAMPLES 1.5: *Factorising Quadratics*

(i) x^2+5x+6

- Look for two numbers that multiply to give 6, and add to give 5:

$$2\times 3=6 \text{ and } 2+3=5$$

- Split the "x"-term into two:

$$x^2+2x+3x+6$$

- Factorise the first pair of terms, and the second pair:

$$x(x+2)+3(x+2)$$

- $(x+2)$ is a factor of both terms so we can rewrite this as:
$$(x+3)(x+2)$$
- So we have:
$$x^2+5x+6=(x+3)(x+2)$$

(ii) y^2-y-12
In this example the two numbers we need are 3 and -4, because $3\times(-4)=-12$ and $3+(-4)=-1$. Hence:
$$\begin{aligned} y^2-y-12 &= y^2+3y-4y-12 \\ &= y(y+3)-4(y+3) \\ &= (y-4)(y+3) \end{aligned}$$

(iii) $2x^2-5x-12$
This example is slightly different because the coefficient of x^2 is not 1.
- Start by multiplying together the coefficient of x^2 and the constant:
$$2\times(-12)=-24$$
- Find two numbers that multiply to give -24, and add to give -5.
$$3\times(-8)=-24 \text{ and } 3+(-8)=-5$$
- Proceed as before:
$$\begin{aligned} 2x^2-5x-12 &= 2x^2+3x-8x-12 \\ &= x(2x+3)-4(2x+3) \\ &= (x-4)(2x+3) \end{aligned}$$

(iv) x^2+x-1
The method doesn't work for this example, because we can't see any numbers that multiply to give -1, but add to give 1. (In fact there is a pair of numbers that does so, but they are not integers so we are unlikely to find them.)

(v) x^2-49
The two numbers must multiply to give -49 and add to give zero. So they are 7 and -7:
$$\begin{aligned} x^2-49 &= x^2+7x-7x-49 \\ &= x(x+7)-7(x+7) \\ &= (x-7)(x+7) \end{aligned}$$

The last example is a special case of the result known as "the difference of two squares". If a and b are any two numbers:
$$a^2-b^2=(a-b)(a+b)$$

Exercises 1.4: Use the method above (if possible) to factorise the following quadratics:

(1) x^2+4x+3
(2) $y^2+10-7y$
(3) $2x^2+7x+3$
(4) $z^2+2z-15$
(5) $4x^2-9$
(6) $y^2-10y+25$
(7) x^2+3x+1

1.6. Rational Numbers, Irrational Numbers, and Square Roots

A *rational* number is a number that can be written in the form $\frac{p}{q}$ where p and q are integers. An *irrational* number is a number that is not rational. It can be shown that if a number can be written as a terminating decimal (such as 1.32) or a recurring decimal (such as 3.7425252525...) then it is rational. Any decimal that does not terminate or recur is irrational.

EXAMPLES 1.6: *Rational and Irrational Numbers*

(i) 3.25 is rational because $3.25 = 3\frac{1}{4} = \frac{13}{4}$.

(ii) -8 is rational because $-8 = \frac{-8}{1}$. Obviously, all integers are rational.

(iii) To show that 0.12121212... is rational check on a calculator that it is equal to $\frac{4}{33}$.

(iv) $\sqrt{2} = 1.41421356237...$ is irrational.

Most, but not all, square roots are irrational:

EXAMPLES 1.7: *Square Roots*

(i) (Using a calculator) $\sqrt{5} = 2.2360679774...$ and $\sqrt{12} = 3.4641016151...$

(ii) $5^2 = 25$, so $\sqrt{25} = 5$

(iii) $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, so $\sqrt{\frac{4}{9}} = \frac{2}{3}$

Rules for Square Roots:

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \text{and} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

EXAMPLES 1.8: *Using the rules to manipulate expressions involving square roots*

(i) $\sqrt{2} \times \sqrt{50} = \sqrt{2 \times 50} = \sqrt{100} = 10$

(ii) $\sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$

(iii) $\frac{\sqrt{98}}{\sqrt{8}} = \sqrt{\frac{98}{8}} = \sqrt{\frac{49}{4}} = \frac{\sqrt{49}}{\sqrt{4}} = \frac{7}{2}$

(iv) $\frac{-2+\sqrt{20}}{2} = -1 + \frac{\sqrt{20}}{2} = -1 + \frac{\sqrt{5}\sqrt{4}}{2} = -1 + \sqrt{5}$

(v) $\frac{8}{\sqrt{2}} = \frac{8\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$

(vi) $\frac{\sqrt{27y}}{\sqrt{3y}} = \sqrt{\frac{27y}{3y}} = \sqrt{9} = 3$

(vii) $\sqrt{x^3y}\sqrt{4xy} = \sqrt{x^3y \times 4xy} = \sqrt{4x^4y^2} = \sqrt{4}\sqrt{x^4}\sqrt{y^2} = 2x^2y$

EXERCISES 1.5: **Square Roots**

(1) Show that: (a) $\sqrt{2} \times \sqrt{18} = 6$ (b) $\sqrt{245} = 7\sqrt{5}$ (c) $\frac{15}{\sqrt{3}} = 5\sqrt{3}$

(2) Simplify: (a) $\frac{\sqrt{45}}{3}$ (b) $\sqrt{2x^3} \times \sqrt{8x}$ (c) $\sqrt{2x^3} \div \sqrt{8x}$ (d) $\frac{1}{3}\sqrt{18y^2}$

Further reading and exercises

- *Jacques* §1.4 has lots more practice of algebra. If you have had any difficulty with the work so far, you should work through it before proceeding.

2. Indices and Logarithms

2.1. Indices

We know that x^3 means $x \times x \times x$. More generally, if n is a positive integer, x^n means "x multiplied by itself n times". We say that x is *raised to the power* n. Alternatively, n may be described as the *index* of x in the expression x^n.

EXAMPLES 2.1:

(i) $5^4 \times 5^3 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^7$.

(ii) $\dfrac{x^5}{x^2} = \dfrac{x \times x \times x \times x \times x}{x \times x} = x \times x \times x = x^3$.

(iii) $\left(y^3\right)^2 = y^3 \times y^3 = y^6$.

Each of the above examples is a special case of the general rules:

- $a^m \times a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{m \times n}$

Now, a^n also has a meaning when n is zero, or negative, or a fraction. Think about the second rule above. If $m = n$, this rule says:

$$a^0 = \frac{a^n}{a^n} = 1$$

If $m = 0$ the rule says:

$$a^{-n} = \frac{1}{a^n}$$

Then think about the third rule. If, for example, $m = \frac{1}{2}$ and $n = 2$, this rule says:

$$\left(a^{\frac{1}{2}}\right)^2 = a$$

which means that

$$a^{\frac{1}{2}} = \sqrt{a}$$

Similarly $a^{\frac{1}{3}}$ is the cube root of a, and more generally $a^{\frac{1}{n}}$ is the n^{th} root of a:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Applying the third rule above, we find for more general fractions:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

We can summarize the rules for zero, negative, and fractional powers:

- $a^0 = 1$ (if $a \neq 0$)
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

There are two other useful rules, which may be obvious to you. If not, check them using some particular examples:

- $a^n b^n = (ab)^n$ and
- $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

EXAMPLES 2.2: *Using the Rules for Indices*

(i) $3^2 \times 3^3 = 3^5 = 243$

(ii) $\left(5^2\right)^{\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

(iii) $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$

(iv) $36^{-\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^{-3} = 6^{-3} = \dfrac{1}{6^3} = \dfrac{1}{216}$

(v) $\left(3\tfrac{3}{8}\right)^{\frac{2}{3}} = \left(\dfrac{27}{8}\right)^{\frac{2}{3}} = \dfrac{27^{\frac{2}{3}}}{8^{\frac{2}{3}}} = \dfrac{\left(27^{\frac{1}{3}}\right)^2}{\left(8^{\frac{1}{3}}\right)^2} = \dfrac{3^2}{2^2} = \dfrac{9}{4}$

2.2. Logarithms

You can think of *logarithm* as another word for index or power. To define a logarithm we first choose a particular *base*. Your calculator probably uses base 10, but we can take any positive integer, a. Now take any positive number, x.

The logarithm of x to the base a is:
the power of a that is equal to x.
If $x = a^n$ then $\log_a x = n$

In fact the statement: $\log_a x = n$ is simply another way of saying: $x = a^n$. Note that, since a^n is positive for all values of n, there is no such thing as the log of zero or a negative number.

EXAMPLES 2.3:

(i) Since we know $2^5 = 32$, we can say that the log of 32 to the base 2 is 5: $\log_2 32 = 5$

(ii) From $3^4 = 81$ we can say $\log_3 81 = 4$

(iii) From $10^{-2} = 0.01$ we can say $\log_{10} 0.01 = -2$

(iv) From $9^{\frac{1}{2}} = 3$ we can say $\log_9 3 = 0.5$

(v) From $a^0 = 1$, we can say that the log of 1 to *any* base is zero: $\log_a 1 = 0$

(vi) From $a^1 = a$, we can say that for any base a, the log of a is 1: $\log_a a = 1$

Except for easy examples like these, you cannot calculate logarithms of particular numbers in your head. For example, if you wanted to know the logarithm to base 10 of 3.4, you would need to find out what power of 10 is equal to 3.4, which is not easy. So instead, you can use your calculator. Check the following examples of logs to base 10:

EXAMPLES 2.4: Using a calculator we find that (correct to 5 decimal places):

(i) $\log_{10} 3.4 = 0.53148$ (ii) $\log_{10} 125 = 2.09691$ (iii) $\log_{10} 0.07 = -1.15490$

There is a way of calculating logs to other bases, using logs to base 10. But the only other base that you really need is the special base e, which we will meet later.

2.3. Rules for Logarithms

Since logarithms are powers, or indices, there are rules for logarithms which are derived from the rules for indices in section 2.1:

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^b = b \log_a x$
- $\log_a a = 1$
- $\log_a 1 = 0$

To see where the first rule comes from, suppose: $m = \log_a x$ and $n = \log_a y$
This is equivalent to: $x = a^m$ and $y = a^n$
Using the first rule for indices: $xy = a^m a^n = a^{m+n}$
But this means that: $\log_a xy = m + n = \log_a x + \log_a y$
which is the first rule for logs.
You could try proving the other rules similarly.

Before electronic calculators were available, printed tables of logs were used calculate, for example, $14.58 \div 0.3456$. You could find the log of each number in the tables, then (applying the second rule) subtract them, and use the tables to find the "anti-log" of the answer.

Examples 2.5: *Using the Rules for Logarithms*

(i) Express $2\log_a 5 + \frac{1}{3}\log_a 8$ as a single logarithm.

$$\begin{aligned} 2\log_a 5 + \tfrac{1}{3}\log_a 8 &= \log_a 5^2 + \log_a 8^{\frac{1}{3}} = \log_a 25 + \log_a 2 \\ &= \log_a 50 \end{aligned}$$

(ii) Express $\log_a \left(\frac{x^2}{y^3}\right)$ in terms of $\log x$ and $\log y$.

$$\begin{aligned} \log_a \left(\frac{x^2}{y^3}\right) &= \log_a x^2 - \log_a y^3 \\ &= 2\log_a x - 3\log_a y \end{aligned}$$

Exercises 1.6: **Indices and Logarithms**

(1) Evaluate (without a calculator):
(a) $64^{\frac{2}{3}}$ (b) $\log_2 64$ (c) $\log_{10} 1000$ (d) $4^{130} \div 4^{131}$

(2) Simplify: (a) $2x^5 \times x^6$ (b) $\dfrac{(xy)^2}{x^3y^2}$ (c) $\log_{10}(xy) - \log_{10} x$ (d) $\log_{10}(x^3) \div \log_{10} x$

(3) Simplify: (a) $\left(3\sqrt{ab}\right)^6$ (b) $\log_{10} a^2 + \frac{1}{3}\log_{10} b - 2\log_{10} ab$

Further reading and exercises

- *Jacques* §2.3 covers all the material in section 2, and provides more exercises.

3. Solving Equations

3.1. Linear Equations

Suppose we have an equation:

$$5(x-6) = x+2$$

Solving this equation means finding the value of x that makes the equation true. (Some equations have several, or many, solutions; this one has only one.)

To solve this sort of equation, we manipulate it by "doing the same thing to both sides." The aim is to get the variable x on one side, and everything else on the other.

EXAMPLES 3.1: Solve the following equations:

(i) $5(x-6) = x+2$

$$\begin{array}{lrcl}
\text{Remove brackets:} & 5x-30 & = & x+2 \\
-x \text{ from both sides:} & 5x-x-30 & = & x-x+2 \\
\text{Collect terms:} & 4x-30 & = & 2 \\
+30 \text{ to both sides:} & 4x & = & 32 \\
\div \text{ both sides by 4:} & x & = & 8
\end{array}$$

(ii) $\dfrac{5-x}{3} + 1 = 2x+4$

Here it is a good idea to remove the fraction first:

$$\begin{array}{lrcl}
\times \text{ all terms by 3:} & 5-x+3 & = & 6x+12 \\
\text{Collect terms:} & 8-x & = & 6x+12 \\
-6x \text{ from both sides:} & 8-7x & = & 12 \\
-8 \text{ from both sides:} & -7x & = & 4 \\
\div \text{ both sides by } -7\text{:} & x & = & -\frac{4}{7}
\end{array}$$

(iii) $\dfrac{5x}{2x-9} = 1$

Again, remove the fraction first:

$$\begin{array}{lrcl}
\times \text{ by } (2x-9)\text{:} & 5x & = & 2x-9 \\
-2x \text{ from both sides:} & 3x & = & -9 \\
\div \text{ both sides by 3:} & x & = & -3
\end{array}$$

All of these are linear equations: once we have removed the brackets and fractions, each term is either an x-term or a constant.

EXERCISES 1.7: Solve the following equations:

(1) $5x+4 = 19$

(2) $2(4-y) = y+17$

(3) $\frac{2x+1}{5} + x - 3 = 0$

(4) $2 - \frac{4-z}{z} = 7$

(5) $\frac{1}{4}(3a+5) = \frac{3}{2}(a+1)$

3.2. Equations involving Parameters

Suppose x satisfies the equation: $5(x-a) = 3x+1$

Here a is a *parameter*: a letter representing an unspecified number. The solution of the equation will depend on the value of a. For example, you can check that if $a = 1$, the solution is $x = 3$, and if $a = 2$ the solution is $x = 5.5$.

Without knowing the value of a, we can still solve the equation for x, to find out exactly how x depends on a. As before, we manipulate the equation to get x on one side and everything else on the other:

$$\begin{aligned} 5x - 5a &= 3x + 1 \\ 2x - 5a &= 1 \\ 2x &= 5a + 1 \\ x &= \frac{5a+1}{2} \end{aligned}$$

We have obtained the solution for x in terms of the parameter a.

EXERCISES 1.8: **Equations involving parameters**

(1) Solve the equation $ax + 4 = 10$ for x.

(2) Solve the equation $\frac{1}{2}y + 5b = 3b$ for y.

(3) Solve the equation $2z - a = b$ for z.

3.3. Changing the Subject of a Formula

$V = \pi r^2 h$ is the formula for the volume of a cylinder with radius r and height h - so if you know r and h, you can calculate V. We could rearrange the formula to *make r the subject*:

$$\begin{aligned} &\text{Write the equation as:} & \pi r^2 h &= V \\ &\text{Divide by } \pi h\text{:} & r^2 &= \frac{V}{\pi h} \\ &\text{Square root both sides:} & r &= \sqrt{\frac{V}{\pi h}} \end{aligned}$$

This gives us a formula for r in terms of V and h. The procedure is exactly the same as solving the equation for r.

EXERCISES 1.9: **Formulae and Equations**

(1) Make t the subject of the formula $v = u + at$

(2) Make a the subject of the formula $c = \sqrt{a^2 + b^2}$

(3) When the price of an umbrella is p, and daily rainfall is r, the number of umbrellas sold is given by the formula: $n = 200r - \frac{p}{6}$. Find the formula for the price in terms of the rainfall and the number sold.

(4) If a firm that manufuctures widgets has m machines and employs n workers, the number of widgets it produces each day is given by the formula $W = m^2(n-3)$. Find a formula for the number of workers it needs, if it has m machines and wants to produce W widgets.

3.4. Quadratic Equations

A quadratic equation is one that, once brackets and fractions have removed, contains terms in x^2, as well as (possibly) x-terms and constants. A quadratic equation can be rearranged to have the form:

$$ax^2 + bx + c = 0$$

where a, b and c are numbers and $a \neq 0$.

A simple quadratic equation is:

$$x^2 = 25$$

You can see immediately that $x = 5$ is a solution, but note that $x = -5$ satisfies the equation too. There are two solutions:

$$x = 5 \quad \text{and} \quad x = -5$$

Quadratic equations have either two solutions, or one solution, or no solutions. The solutions are also known as the *roots* of the equation. There are two general methods for solving quadratics; we will apply them to the example:

$$x^2 + 5x + 6 = 0$$

Method 1: Quadratic Factorisation
We saw in section 1.5 that the quadratic polynomial $x^2 + 5x + 6$ can be factorised, so we can write the equation as:

$$(x + 3)(x + 2) = 0$$

But if the product of two expressions is zero, this means that one of them must be zero, so we can say:

$$\begin{aligned} \text{either } x + 3 = 0 &\quad \Rightarrow \quad x = -3 \\ \text{or } x + 2 = 0 &\quad \Rightarrow \quad x = -2 \end{aligned}$$

The equation has two solutions, -3 and -2. You can check that these are solutions by substituting them back into the original equation.

Method 2: The Quadratic Formula
If the equation $ax^2 + bx + c = 0$ can't be factorised (or if it can, but you can't see how) you can use[1]:

$$\boxed{\text{The Quadratic Formula:} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The notation $\pm$ indicates that an expression may take either a positive or negative value. So this is a formula for the two solutions $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

In the equation $x^2 + 5x + 6 = 0$, $a = 1$, $b = 5$ and $c = 6$. The quadratic formula gives us:

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 6}}{2 \times 1} \\ &= \frac{-5 \pm \sqrt{25 - 24}}{2} \\ &= \frac{-5 \pm 1}{2} \end{aligned}$$

[1] *Antony & Biggs* §2.4 explains where the formula comes from.

So the two solutions are:

$$x = \frac{-5+1}{2} = -2 \text{ and } x = \frac{-5-1}{2} = -3$$

Note that in the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, $b^2 - 4ac$ could turn out to be zero, in which case there is only one solution. Or it could be negative, in which case there are no solutions since we can't take the square root of a negative number.

EXAMPLES 3.2: Solve, if possible, the following quadratic equations.

(i) $x^2 + 3x - 10 = 0$
Factorise:

$$(x+5)(x-2) = 0$$
$$\Rightarrow x = -5 \text{ or } x = 2$$

(ii) $x(7-2x) = 6$
First, rearrange the equation to get it into the usual form:

$$\begin{aligned} 7x - 2x^2 &= 6 \\ -2x^2 + 7x - 6 &= 0 \\ 2x^2 - 7x + 6 &= 0 \end{aligned}$$

Now, we can factorise, to obtain:

$$(2x-3)(x-2) = 0$$

$$\begin{aligned} \text{either } 2x - 3 = 0 \quad &\Rightarrow \quad x = \tfrac{3}{2} \\ \text{or } x - 2 = 0 \quad &\Rightarrow \quad x = 2 \end{aligned}$$

The solutions are $x = \frac{3}{2}$ and $x = 2$.

(iii) $y^2 + 4y + 4 = 0$
Factorise:

$$\begin{aligned} (y+2)(y+2) &= 0 \\ \Longrightarrow y + 2 &= 0 \quad \Rightarrow y = -2 \end{aligned}$$

Therefore $y = -2$ is the only solution. (Or we sometimes say that the equation has a *repeated root* – the two solutions are the same.)

(iv) $x^2 + x - 1 = 0$
In section 1.5 we couldn't find the factors for this example. So apply the formula, putting $a = 1$, $b = 1$, $c = -1$:

$$x = \frac{-1 \pm \sqrt{1-(-4)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

The two solutions, correct to 3 decimal places, are:

$$x = \tfrac{-1+\sqrt{5}}{2} = 0.618 \quad \text{and} \quad x = \tfrac{-1-\sqrt{5}}{2} = -1.618$$

Note that this means that the factors are, approximately, $(x - 0.618)$ and $(x + 1.618)$.

(v) $2z^2 + 2z + 5 = 0$

Applying the formula gives: $z = \dfrac{-2 \pm \sqrt{-36}}{4}$

So there are no solutions, because this contains the square root of a negative number.

(vi) $6x^2 + 2kx = 0$ (solve for x, treating k as a parameter)

Factorising:

$$2x(3x + k) = 0$$

$$\begin{aligned} \text{either } 2x = 0 \quad &\Rightarrow \quad x = 0 \\ \text{or } 3x + k = 0 \quad &\Rightarrow \quad x = -\frac{k}{3} \end{aligned}$$

EXERCISES 1.10: Solve the following quadratic equations, where possible:

(1) $x^2 + 3x - 13 = 0$

(2) $4y^2 + 9 = 12y$

(3) $3z^2 - 2z - 8 = 0$

(4) $7x - 2 = 2x^2$

(5) $y^2 + 3y + 8 = 0$

(6) $x(2x - 1) = 2(3x - 2)$

(7) $x^2 - 6kx + 9k^2 = 0$ (where k is a parameter)

(8) $y^2 - 2my + 1 = 0$ (where m is a parameter)
Are there any values of m for which this equation has no solution?

3.5. Equations involving Indices

EXAMPLES 3.3:

(i) $7^{2x+1} = 8$

Here the variable we want to find, x, appears in a power.
This type of equation can by solved by *taking logs of both sides*:

$$\begin{aligned} \log_{10}\left(7^{2x+1}\right) &= \log_{10}(8) \\ (2x+1)\log_{10} 7 &= \log_{10} 8 \\ 2x + 1 &= \frac{\log_{10} 8}{\log_{10} 7} = 1.0686 \\ 2x &= 0.0686 \\ x &= 0.0343 \end{aligned}$$

(ii) $(2x)^{0.65} + 1 = 6$

We can use the rules for indices to manipulate this equation:

$$\begin{array}{lrcl}
\text{Subtract 1 from both sides:} & (2x)^{0.65} & = & 5 \\
\text{Raise both sides to the power } \frac{1}{0.65}\text{:} & \left((2x)^{0.65}\right)^{\frac{1}{0.65}} & = & 5^{\frac{1}{0.65}} \\
 & 2x & = & 5^{\frac{1}{0.65}} = 11.894 \\
\text{Divide by 2:} & x & = & 5.947
\end{array}$$

3.6. Equations involving Logarithms

Examples 3.4: Solve the following equations:

(i) $\log_5(3x-2) = 2$
From the definition of a logarithm, this equation is equivalent to:

$$3x - 2 = 5^2$$

which can be solved easily:

$$3x - 2 = 25 \Rightarrow x = 9$$

(ii) $10\log_{10}(5x+1) = 17$

$$\begin{array}{rcl}
\Rightarrow \log_{10}(5x+1) & = & 1.7 \\
5x+1 & = & 10^{1.7} = 50.1187 \text{ (correct to 4 decimal places)} \\
x & = & 9.8237
\end{array}$$

Exercises 1.11: Solve the following equations:

(1) $\log_4(2+x) = 2$
(2) $16 = 5^{3t}$
(3) $2 + x^{0.4} = 8$

The remaining questions are a bit harder – skip them if you found this section difficult.

(4) $4.1 + 5x^{0.42} = 7.8$
(5) $6^{x^2-7} = 36$
(6) $\log_2(y^2+4) = 3$
(7) $3^{n+1} = 2^n$
(8) $2\log_{10}(x-2) = \log_{10}(x)$

Further reading and exercises

- For more practice on solving all the types of equation in this section, you could use an A-level pure maths textbook.
- *Jacques* §1.5 gives more detail on *Changing the Subject of a Formula*
- *Jacques* §2.1 and *Anthony & Biggs* §2.4 both cover the Quadratic Formula for *Solving Quadratic Equations*
- *Jacques* §2.3 has more *Equations involving Indices*

4. Simultaneous Equations

So far we have looked at equations involving one variable (such as x). An equation involving two variables, x and y, such as $x + y = 20$, has lots of solutions – there are lots of pairs of numbers x and y that satisfy it (for example $x = 3$ and $y = 17$, or $x = -0.5$ and $y = 20.5$).

But suppose we have two equations and two variables:

$$\begin{aligned} &(1) & x + y &= 20 \\ &(2) & 3x &= 2y - 5 \end{aligned}$$

There is just one pair of numbers x and y that satisfy both equations.

Solving a pair of simultaneous equations means finding the pair(s) of values that satisfy both equations. There are two approaches; in both the aim is to eliminate one of the variables, so that you can solve an equation involving one variable only.

Method 1: Substitution

Make one variable the subject of one of the equations (it doesn't matter which), and substitute it in the other equation.

From equation (1): $$x = 20 - y$$

Substitute for x in equation (2): $$3(20 - y) = 2y - 5$$

Solve for y: $$\begin{aligned} 60 - 3y &= 2y - 5 \\ -5y &= -65 \\ y &= 13 \end{aligned}$$

From the equation in the first step: $$x = 20 - 13 = 7$$

The solution is $x = 7, y = 13$.

Method 2: Elimination

Rearrange the equations so that you can add or subtract them to eliminate one of the variables.

Write the equations as: $$\begin{aligned} x + y &= 20 \\ 3x - 2y &= -5 \end{aligned}$$

Multiply the first one by 2: $$\begin{aligned} 2x + 2y &= 40 \\ 3x - 2y &= -5 \end{aligned}$$

Add the equations together: $$5x = 35 \quad \Rightarrow x = 7$$

Substitute back in equation (1): $$7 + y = 20 \quad \Rightarrow y = 13$$

EXAMPLES 4.1: *Simultaneous Equations*

(i) Solve the equations $3x + 5y = 12$ and $2x - 6y = -20$

Multiply the first equation by 2 and the second one by 3:

$$\begin{array}{lrcll} & 6x + 10y & = & 24 & \\ & 6x - 18y & = & \text{-60} & \\ \text{Subtract:} & 28y & = & 84 & \Rightarrow y = 3 \\ \text{Substitute back in the } 2^{nd} \text{ equation:} & 2x - 18 & = & -20 & \Rightarrow x = -1 \end{array}$$

(ii) Solve the equations $x + y = 3$ and $x^2 + 2y^2 = 18$

Here the first equation is linear but the second is quadratic.
Use the linear equation for a substitution:

$$\begin{aligned} x &= 3 - y \\ \Rightarrow (3 - y)^2 + 2y^2 &= 18 \\ 9 - 6y + y^2 + 2y^2 &= 18 \\ 3y^2 - 6y - 9 &= 0 \\ y^2 - 2y - 3 &= 0 \end{aligned}$$

Solving this quadratic equation gives two solutions for y:

$$y = 3 \text{ or } y = -1$$

Now find the corresponding values of x using the linear equation: when $y = 3, x = 0$ and when $y = -1, x = 4$. So there are two solutions:

$$x = 0, y = 3 \quad \text{and} \quad x = 4, y = -1$$

(iii) Solve the equations $x + y + z = 6$, $y = 2x$, and $2y + z = 7$

Here we have three equations, and three variables. We use the same methods, to eliminate first one variable, then another.
Use the second equation to eliminate y from both of the others:

$$\begin{aligned} x + 2x + z = 6 \Rightarrow 3x + z &= 6 \\ 4x + z &= 7 \end{aligned}$$

Eliminate z by subtracting: $x = 1$
Work out z from $4x + z = 7$: $z = 3$
Work out y from $y = 2x$: $y = 2$
The solution is $x = 1, y = 2, z = 3$.

EXERCISES 1.12: Solve the following sets of simultaneous equations:

(1) $2x = 1 - y$ and $3x + 4y + 6 = 0$

(2) $2z + 3t = -0.5$ and $2t - 3z = 10.5$

(3) $x + y = a$ and $x = 2y$ for x and y, in terms of the parameter a.

(4) $a = 2b$, $a + b + c = 12$ and $2b - c = 13$

(5) $x - y = 2$ and $x^2 = 4 - 3y^2$

Further reading and exercises

- *Jacques* §1.2 covers *Simultaneous Linear Equations* thoroughly.

5. Inequalities and Absolute Value

5.1. Inequalities

$$2x + 1 \le 6$$

is an example of an inequality. Solving the inequality means "finding the set of values of x that make the inequality true." This can be done very similarly to solving an equation:

$$\begin{aligned} 2x + 1 &\le 6 \\ 2x &\le 5 \\ x &\le 2.5 \end{aligned}$$

Thus, all values of x less than or equal to 2.5 satisfy the inequality.

> When manipulating inequalities you can add anything to both sides, or subtract anything, and you can multiply or divide both sides by a positive number. But if you multiply or divide both sides by a negative number you must reverse the inequality sign.

To see why you have to reverse the inequality sign, think about the inequality:

	5	$<$	8	(which is true)
If you multiply both sides by 2, you get:	10	$<$	16	(also true)
But if you just multiplied both sides by -2, you would get:	-10	$<$	-16	(NOT true)
Instead we reverse the sign when multiplying by -2, to obtain:	-10	$>$	-16	(true)

Examples 5.1: Solve the following inequalities:

(i) $3(x + 2) > x - 4$

$$\begin{aligned} 3x + 6 &> x - 4 \\ 2x &> -10 \\ x &> -5 \end{aligned}$$

(ii) $1 - 5y \le -9$

$$\begin{aligned} -5y &\le -10 \\ y &\ge 2 \end{aligned}$$

5.2. Absolute Value

The absolute value, or *modulus*, of x is the positive number which has the same "magnitude" as x. It is denoted by $|x|$. For example, if $x = -6$, $|x| = 6$ and if $y = 7$, $|y| = 7$.

$$|x| = x \text{ if } x \ge 0$$
$$|x| = -x \text{ if } x < 0$$

Examples 5.2: *Solving equations and inequalities involving absolute values*

(i) Find the values of x satisfying $|x + 3| = 5$.

$$|x + 3| = 5 \Rightarrow x + 3 = \pm 5$$

$$\begin{aligned} \text{Either:}\quad x+3 &= 5 &&\Rightarrow x=2\\ \text{or:}\quad x+3 &= -5 &&\Rightarrow x=-8 \end{aligned}$$

So there are two solutions: $x = 2$ and $x = -8$

(ii) Find the values of y for which $|y| \leq 6$.

$$\begin{aligned} \text{Either:}\quad y &\leq 6 \\ \text{or:}\quad -y &\leq 6 \quad\Rightarrow\quad y \geq -6 \end{aligned}$$

So the solution is: $-6 \leq y \leq 6$

(iii) Find the values of z for which $|z-2| > 4$.

$$\begin{aligned} \text{Either:}\quad z-2 &> 4 \quad \Rightarrow z>6\\ \text{or:}\quad -(z-2) &> 4 \quad \Rightarrow z-2<-4 \quad \Rightarrow z<-2 \end{aligned}$$

So the solution is: $z < -2$ or $z > 6$

5.3. Quadratic Inequalities

EXAMPLES 5.3: Solve the inequalities:

(i) $x^2 - 2x - 15 \leq 0$
Factorise:

$$(x-5)(x+2) \leq 0$$

If the product of two factors is negative, one must be negative and the other positive:

$$\begin{aligned} \text{either:}\quad & x-5 \leq 0 \text{ and } x+2 \geq 0 \quad \Rightarrow -2 \leq x \leq 5\\ \text{or:}\quad & x-5 \geq 0 \text{ and } x+2 \leq 0 \text{ which is impossible.} \end{aligned}$$

So the solution is: $-2 \leq x \leq 5$

(ii) $x^2 - 7x + 6 > 0$

$$\Rightarrow (x-6)(x-1) > 0$$

If the product of two factors is positive, both must be positive, or both negative:

$$\begin{aligned} \text{either:}\quad & x-6>0 \text{ and } x-1>0 \quad \text{which is true if: } x>6\\ \text{or:}\quad & x-6<0 \text{ and } x-1<0 \quad \text{which is true if: } x<1 \end{aligned}$$

So the solution is: $x < 1$ or $x > 6$

EXERCISES 1.13: Solve the following equations and inequalities:

(1) (a) $2x + 1 \geq 7$ (b) $5(3 - y) < 2y + 3$

(2) (a) $|9 - 2x| = 11$ (b) $|1 - 2z| > 2$

(3) $|x + a| < 2$ where a is a parameter, and we know that $0 < a < 2$.

(4) (a) $x^2 - 8x + 12 < 0$ (b) $5x - 2x^2 \leq -3$

Further reading and exercises

- *Jacques* §1.4.1 has a little more on *Inequalities*.
- Refer to an A-level pure maths textbook for more detail and practice.

Solutions to Exercises in Chapter 1

Exercises 1.1:

(1) (a) 29
(b) -8
(c) -15
(d) $\frac{1}{2}$
(e) 64
(f) 3

Exercises 1.2:

(1) (a) $x^3 + 13x - 25$
(b) $2x^2 - 8y$ or $2(x^2 - 4y)$
(2) (a) $3z^2x + 2z - 1$
(b) $7x + 14$ or $7(x+2)$
(3) (a) $\frac{xy}{2}$
(b) $\frac{6y}{x}$
(4) (a) $\frac{x}{4y}$
(b) $20x^3y^4$
(5) (a) y
(b) $\frac{4x^2}{y^3}$
(6) (a) $\frac{10x+3}{12}$
(b) $\frac{2}{x^2-1}$

Exercises 1.3:

(1) (a) $3x(1+2y)$
(b) $y(2y+7)$
(c) $3(2a+b+3c)$
(2) (a) $x^2(3x-10)$
(b) $c(a-b)$
(3) $(x+2)(y+2z)$
(4) $x(2-x)$

Exercises 1.4:

(1) $(x+1)(x+3)$
(2) $(y-5)(y-2)$
(3) $(2x+1)(x+3)$
(4) $(z+5)(z-3)$
(5) $(2x+3)(2x-3)$
(6) $(y-5)^2$
(7) Not possible to split into integer factors.

Exercises 1.5:

(1) (a) $= \sqrt{2 \times 18}$
$= \sqrt{36} = 6$
(b) $= \sqrt{49 \times 5}$
$= 7\sqrt{5}$
(c) $\frac{15}{\sqrt{3}} = \frac{15\sqrt{3}}{3}$
$= 5\sqrt{3}$
(2) (a) $\sqrt{5}$
(b) $4x^2$
(c) $\frac{x}{2}$
(d) $\sqrt{2}y$

Exercises 1.6:

(1) (a) 16
(b) 6
(c) 3
(d) $\frac{1}{4}$
(2) (a) $2x^{11}$
(b) $\frac{1}{x}$
(c) $\log_{10} y$
(d) 3
(3) (a) $(9ab)^3$
(b) $-\frac{5}{3}\log_{10} b$

Exercises 1.7:

(1) $x = 3$
(2) $y = -3$
(3) $x = 2$
(4) $z = -1$
(5) $a = -\frac{1}{3}$

Exercises 1.8:

(1) $x = \frac{6}{a}$
(2) $y = -4b$
(3) $z = \frac{a+b}{2}$

Exercises 1.9:

(1) $t = \frac{v-u}{a}$
(2) $a = \sqrt{c^2 - b^2}$
(3) $p = 1200r - 6n$
(4) $n = \frac{W}{m^2} + 3$

Exercises 1.10:

(1) $x = \frac{-3 \pm \sqrt{61}}{2}$
(2) $y = 1.5$
(3) $z = -\frac{4}{3}, 2$
(4) $x = \frac{7 \pm \sqrt{33}}{4}$
(5) No solutions.
(6) $x = \frac{7 \pm \sqrt{17}}{4}$
(7) $x = 3k$
(8) $y = (m \pm \sqrt{m^2 - 1})$
No solution if $-1 < m < 1$

Exercises 1.11:

(1) $x = 14$
(2) $t = \frac{\log(16)}{3\log(5)} = 0.5742$
(3) $x = 6^{\frac{1}{0.4}} = 88.18$
(4) $x = (0.74)^{\frac{1}{0.42}} = 0.4883$
(5) $x = \pm 3$
(6) $y = \pm 2$
(7) $n = \frac{\log_2 3}{\log_2 \frac{2}{3}} = -2.7095$
(8) $x = 4, x = 1$

Exercises 1.12:

(1) $x = 2$, $y = -3$
(2) $t = 1.5$, $z = -2.5$
(3) $x = \frac{2a}{3}$, $y = \frac{a}{3}$
(4) $a = 10$, $b = 5$, $c = -3$
(5) $(x, y) = (2, 0)$
$(x, y) = (1, -1)$

Exercises 1.13:

(1) (a) $x \geq 3$
(b) $\frac{12}{7} < y$
(2) (a) $x = -1, 10$
(b) $z < -0.5$ or $z > 1.5$
(3) $-2 - a < x < 2 - a$
(4) (a) $2 < x < 6$
(b) $x \geq 3$, $x \leq -0.5$

[2] This Version of Workbook Chapter 1: September 29, 2004

Worksheet 1: Review of Algebra

(1) For a firm, the cost of producing q units of output is $C = 4 + 2q + 0.5q^2$. What is the cost of producing (a) 4 units (b) 1 unit (c) no units?

(2) Evaluate the expression $x^3(y + 7)$ when $x = -2$ and $y = -10$.

(3) Simplify the following algebraic expressions, factorising the answer where possible:
(a) $x(2y+3x-12)-3(2-5xy)-(3x+8xy-6)$ (b) $z(2-3z+5z^2)+3(z^2-z^3-4)$

(4) Simplify: (a) $6a^4b \times 4b \div 8ab^3c$ (b) $\sqrt{3x^3y} \div \sqrt{27xy}$ (c) $(2x^3)^3 \times (xz^2)^4$

(5) Write as a single fraction: (a) $\dfrac{2y}{3x} + \dfrac{4y}{5x}$ (b) $\dfrac{x+1}{4} - \dfrac{2x-1}{3}$

(6) Factorise the following quadratic expressions:
(a) $x^2 - 7x + 12$ (b) $16y^2 - 25$ (c) $3z^2 - 10z - 8$

(7) Evaluate (without using a calculator): (a) $4^{\frac{3}{2}}$ (b) $\log_{10} 100$ (c) $\log_5 125$

(8) Write as a single logarithm: (a) $2\log_a(3x) + \log_a x^2$ (b) $\log_a y - 3\log_a z$

(9) Solve the following equations:
(a) $5(2x-9) = 2(5-3x)$ (b) $1 + \dfrac{6}{y-8} = -1$ (c) $z^{0.4} = 7$ (d) $3^{2t-1} = 4$

(10) Solve these equations for x, in terms of the parameter a:
(a) $ax - 7a = 1$ (b) $5x - a = \dfrac{x}{a}$ (c) $\log_a(2x+5) = 2$

(11) Make Q the subject of: $P = \sqrt{\dfrac{a}{Q^2+b}}$

(12) Solve the equations: (a) $7 - 2x^2 = 5x$ (b) $y^2 + 3y - 0.5 = 0$ (c) $|1 - z| = 5$

(13) Solve the simultaneous equations:
(a) $2x - y = 4$ and $5x = 4y + 13$
(b) $y = x^2 + 1$ and $2y = 3x + 4$

(14) Solve the inequalities: (a) $2y - 7 \leq 3$ (b) $3 - z > 4 + 2z$ (c) $3x^2 < 5x + 2$