C2 Jan 07
Model Solutions

(1) $f(x) = x^3 + 3x^2 + 5$

a) $f'(x) = 3x^2 + 6x$
$f''(x) = 6x + 6$

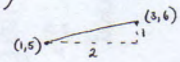
b) $\int_1^2 f(x)\,dx = \int_1^2 x^3 + 3x^2 + 5\,dx$
$= \left[\frac{x^4}{4} + \frac{3x^3}{3} + 5x\right]_1^2$
$= \left[\frac{x^4}{4} + x^3 + 5x\right]_1^2$
$= \left[\left(\frac{2^4}{4} + 2^3 + 5\times2\right) - \left(\frac{1^4}{4} + 1^3 + 5\times1\right)\right]$
$= \left[(22) - \left(\frac{25}{4}\right)\right]$
$= \frac{63}{4}$

(2) a) $(1-2x)^5 = 1 + 5(-2x) + \frac{5\times4}{2!}(-2x)^2 + \frac{5\times4\times3}{3!}(-2x)^3 + \ldots$
$= 1 - 10x + 40x^2 - 80x^3 + \ldots$

b) $(1+x)(1-2x)^5 \approx (1+x)(1 - 10x + 40x^2 + \ldots)$
$\approx (1 - 10x + 40x^2 + x - 10x^2 + \ldots)$
$\approx 1 - 9x + \ldots$
$\approx 1 - 9x$ QED

(3) Midpoint of the line: $M = \left(\frac{-1+3}{2}, \frac{4+6}{2}\right) = (1,5)$ $(= (a,b))$.

Length of radius from $(1,5)$ to $(3,6)$



$r = \sqrt{(3-1)^2 + (6-5)^2}$
$= \sqrt{2^2 + 1^2}$
$= \sqrt{5}$

Equation of circle: $(x-a)^2 + (y-b)^2 = r^2$
$(x-1)^2 + (y-5)^2 = 5$

(4) Method 1:
$5^x = 17$
$\log_5 5^x = \log_5 17$
$x = \log_5 17$
$= 1.76$

Method 2:
$5^x = 17$
$\log 5^x = \log 17$
$x \log 5 = \log 17$
$x = \frac{\log 17}{\log 5} = 1.76$

(5) $f(x) = x^3 + 4x^2 + x - 6$

a) If $(x+2)$ is a factor of $f(x)$, $f(-2) = 0$:
$f(-2) = (-2)^3 + 4\times(-2)^2 + (-2) - 6$
$= -8 + 4\times4 - 2 - 6$
$= -8 + 16 - 2 - 6$
$= 0$ QED

b)
$$\begin{array}{r} x^2 + 2x - 3 \\ x+2\,\overline{)\,x^3 + 4x^2 + x - 6} \\ \underline{x^3 + 2x^2} \\ 2x^2 + x \\ \underline{2x^2 + 4x} \\ -3x - 6 \\ \underline{-3x - 6} \\ 0 \end{array}$$

$f(x) = (x+2)(x^2 + 2x - 3)$
$= (x+2)(x+3)(x-1)$

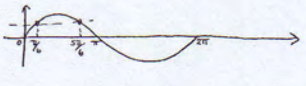
c) $x^3 + 4x^2 + x - 6 = 0$
$\Rightarrow (x+2)(x+3)(x-1) = 0$
$\Rightarrow x = -2,\ x = -3,\ x = 1$

(6) $2\cos^2 x + 1 = 5\sin x$ $\quad 0 \le x \le 2\pi$

Using $\cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$

$2(1 - \sin^2 x) + 1 = 5\sin x$
$2 - 2\sin^2 x + 1 = 5\sin x$
$2\sin^2 x + 5\sin x - 3 = 0$
$(2\sin x - 1)(\sin x + 3) = 0$
$\sin x = \frac{1}{2}$ or $\sin x = -3$
IMPOSSIBLE (NO SOLUTIONS).

$x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \frac{5\pi}{6}$



(7) $y = x(x-1)(x-5) = x(x^2 - 6x + 5)$
$y = x^3 - 6x^2 + 5x$

First region: $\int_0^1 x^3 - 6x^2 + 5x\,dx$
$= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{5x^2}{2}\right]_0^1$
$= \left[\left(\frac{1^4}{4} - \frac{6\times1^3}{3} + \frac{5\times1^2}{2}\right) - (0)\right]$
$= \left(\frac{1}{4} - 2 + \frac{5}{2}\right) - (0)$
$= \frac{3}{4}$

Second region: $\int_1^2 x^3 - 6x^2 + 5x\,dx$
$= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{5x^2}{2}\right]_1^2$
$= \left[\left(\frac{2^4}{4} - \frac{6\times2^3}{3} + \frac{5\times2^2}{2}\right) - \left(\frac{1^4}{4} - \frac{6\times1^3}{3} + \frac{5\times1^2}{2}\right)\right]$
$= (4 - 8 + 10) - \left(\frac{3}{4}\right)$
$= -\frac{11}{4}$

Total area $= \frac{3}{4} + \frac{11}{4} = \frac{14}{4} = \frac{7}{2}$

(8) $C = \frac{1400}{v} + \frac{2v}{7} = 1400v^{-1} + \frac{2}{7}v$.

a) $\frac{dC}{dv} = -1400v^{-2} + \frac{2}{7}$
$= \frac{-1400}{v^2} + \frac{2}{7}$

At min., $\frac{dC}{dv} = 0 \Rightarrow \frac{-1400}{v^2} + \frac{2}{7} = 0$
$\Rightarrow \frac{1400}{v^2} = \frac{2}{7}$
$\Rightarrow 1400 = \frac{2v^2}{7} \Rightarrow 4900 = v^2$
$\Rightarrow v = 70.$

b) $\frac{d^2C}{dv^2} = 2800v^{-3} = \frac{2800}{v^3}$
When $v = 70$, $\frac{2800}{v^3} = \frac{2800}{70^3} = \frac{2}{245} > 0$ $\therefore$ this is a minim
QED

c) Minimum cost, $C = \frac{1400}{70} + \frac{2\times70}{7} = 20 + 20 = £40$

(9)
a) Cosine Rule: $q^2 = p^2 + r^2 - 2pr\cos Q$
$(6\sqrt{3})^2 = 6^2 + 6^2 - 2\times6\times6\cos Q$
$108 = 36 + 36 - 72\cos Q$
$36 = -72\cos Q$
$\Rightarrow -0.5 = \cos Q$
$\Rightarrow Q = \frac{2\pi}{3}$

b) Area (sector) $= \frac{1}{2}r^2\theta = \frac{1}{2}\times6^2\times\frac{2\pi}{3} = 12\pi$ QED

c) Area triangle PQR $= \frac{1}{2}pr\sin Q = \frac{1}{2}\times6\times6\sin\left(\frac{2\pi}{3}\right) = 9\sqrt{3}$

d) Area segment $= 12\pi - 9\sqrt{3} = 22.1\ m^2$.

e) Length of segment PSR $= r\theta = 6\times\frac{2\pi}{3} = 4\pi$

Perimeter of patio $= 6 + 6 + 4\pi = 12 + 4\pi \approx 24.6\ m$

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a) $S_n = a + ar + ar^2 + \dots + ar^{n-1}$

$rS_n = ar + ar^2 + ar^3 + \dots + ar^n$

$S_n - rS_n = (a + ar + ar^2 + \dots + ar^{n-1}) - (ar + ar^2 + ar^3 + \dots + ar^n)$

$= a - ar^n$

$S_n(1-r) = a(1-r^n)$

$S_n = \frac{a(1-r^n)}{1-r}$ QED

b) $\sum_{k=1}^{10} 100(2^k) = 100(2^1) + 100(2^2) + 100(2^3) + \dots + 100(2^{10})$

$= 100(2 + 2\times2 + 2\times2^2 + 2\times2^3 + \dots + 2\times2^9)$

In the bracket is a geometric series where $a = 2$ and $r = 2$:

$S_{10} = \frac{a(1-r^n)}{1-r} = \frac{2(1-2^{10})}{1-2} = 2046$

So $\sum_{k=1}^{10} 100(2^k) = 100 \times 2046 = 204600$

c) $\frac{5}{6} \xrightarrow{\times 1/3} + \frac{5}{18} \xrightarrow{\times 1/3} + \frac{5}{54}$ $a = 5/6$, $r = 1/3$

$S_\infty = \frac{a}{1-r} = \frac{5/6}{1-1/3} = \frac{5/6}{2/3} = 5/4$

d) $-1 < r < 1$