C2 June 2006

Model Solutions

① $(2+x)^6 = \left[2\left(1+\frac{x}{2}\right)\right]^6 = 2^6\left(1+\frac{x}{2}\right)^6 = 64\left(1+\frac{x}{2}\right)^6$
$= 64\left(1+6\left(\frac{x}{2}\right)+\frac{6\times5}{2!}\left(\frac{x}{2}\right)^2+\ldots\right) = 64\left(1+3x+\frac{15x^2}{4}+\ldots\right)$
$= \underline{64+192x+240x^2+\ldots}$

② $\int_1^2 3x^2+5+\frac{4}{x^2}\,dx = \int_1^2 3x^2+5+4x^{-2}\,dx$
$= \left[\frac{3x^3}{3}+5x+\frac{4x^{-1}}{-1}\right]_1^2$
$= \left[x^3+5x-\frac{4}{x}\right]_1^2$
$= \left[\left(2^3+5(2)-\frac{4}{2}\right)-\left(1^3+5(1)-\frac{4}{1}\right)\right]$
$= (16)-(2)$
$= \underline{14}$

③ i) $\log_6 36 = \log_6 6^2 = 2\log_6 6 = \underline{2}$

ii) $2\log_a 3 + \log_a 11 = \log_a 3^2 + \log_a 11 = \log_a 9 + \log_a 11$
$= \log_a(9\times11) = \underline{\log_a 99}$

④ $f(x) = 2x^3+3x^2-29x-60$

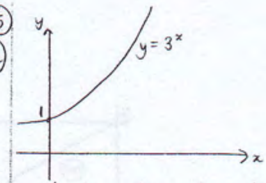
a) $f(-2) = 2(-2)^3+3(-2)^2-29(-2)-60$
$= -16+12+58-60 = \underline{-6}$

b) $f(-3) = 2(-3)^3+3(-3)^2-29(-3)-60$
$= -54+27+87-60 = 0$ QED

c)
$$\begin{array}{r} 2x^2-3x-20 \\ x+3\,\overline{)\,2x^3+3x^2-29x-60} \\ \underline{2x^3+6x^2} \\ -3x^2-29x \\ \underline{-3x^2-9x} \\ -20x-60 \\ \underline{-20x-60} \\ 0 \end{array}$$

$f(x) = (x+3)(2x^2-3x-20)$
$= \underline{(x+3)(2x+5)(x-4)}$

⑤ a)



b)

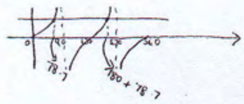
x	0	0.2	0.4	0.6	0.8	1
3^x	1	1.246	1.552	1.933	2.408	3

c) There are 5 strips from the table, so $n=5$, $h=0.2$.

Area $\approx \frac{1}{2}\times0.2\times\left\{1+3+2(1.246+1.552+1.933+2.408)\right\}$
$\approx 0.1\times\{4+2(7.139)\}$
$\approx \underline{1.8278}$

⑥ a) $\sin\theta = 5\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = 5 \Rightarrow \underline{\tan\theta = 5}$

b) $\sin\theta = 5\cos\theta$ $\quad 0\le\theta\le360°$
$\tan\theta = 5$
$\theta = \tan^{-1}(5)$
$= \underline{78.7°},\ \underline{258.7°}$



⑦ a) Line through P and Q is perpendicular to $y=3x-4$, ∴ has gradient $-\frac{1}{3}$.

$y-y_1 = m(x-x_1)$
$y-2 = -\frac{1}{3}(x-2)$
$3(y-2) = -1(x-2)$
$3y-6 = -x+2$
$\underline{3y = 8-x}$

b) Q is the point $(x,1)$
$3\times1 = 8-x \Rightarrow x = 8-3 = 5.$ QED

c) Length of radius $PQ = \sqrt{(2-1)^2+(5-2)^2} = \sqrt{1^2+3^2} = \sqrt{10}$.

Equation of circle $(x-a)^2+(y-b)^2 = r^2$
$\underline{(x-5)^2+(y-1)^2 = 10}$

⑧ a) Arc length $= r\theta = 2.12\times0.65 = 1.378 = \underline{1.38\text{m}}$

b) Area sector $= \frac{1}{2}r^2\theta = \frac{1}{2}\times2.12^2\times0.65 = 1.46068 = \underline{1.46\text{m}^2}$

c) Angle CAD $= \frac{\pi}{2}-0.65 = \underline{0.92^c}$

d) Area triangle CAD $= \frac{1}{2}\times2.12\times1.86\times\sin0.92 = 1.5696$

Area cross-section $= 1.46+1.57 = \underline{3.03\text{m}^2}$

⑨ a) Second term $= ar = 4$
$S_\infty = \frac{a}{1-r} = 25$

If $ar=4 \Rightarrow a = \frac{4}{r}$
S_∞: $\frac{a}{1-r} = 25 \Rightarrow a = 25(1-r)$
$\Rightarrow \frac{4}{r} = 25(1-r)$
$\Rightarrow 4 = 25r(1-r)$
$\Rightarrow 4 = 25r-25r^2$
$\Rightarrow 25r^2-25r+4 = 0$ QED

b) $25r^2-25r+4=0 \Rightarrow (5r-4)(5r-1)=0$
$\Rightarrow \underline{r=\frac{4}{5}}$ or $\underline{r=\frac{1}{5}}$

c) If $r=\frac{4}{5}$, $a = \frac{4}{r} = \frac{4}{4/5} = \underline{5}$
If $r=\frac{1}{5}$, $a = \frac{4}{1/5} = \underline{20}$

d) $S_n = \frac{a(1-r^n)}{1-r}$, $\frac{a}{1-r} = S_\infty = 25$ so $S_n = 25(1-r^n)$ QED

e) $S_n > 24$ $\quad r=\frac{4}{5}, a=5$ $\quad S_n = \frac{a(1-r^n)}{1-r} = 25(1-r^n)$
$25(1-r^n) > 24$
$1-\left(\frac{4}{5}\right)^n > \frac{24}{25}$
$-\left(\frac{4}{5}\right)^n > -\frac{1}{25} \Rightarrow \frac{1}{25} > \left(\frac{4}{5}\right)^n$

Method 1:
$\log_{4/5}\left(\frac{1}{25}\right) > \log_{4/5}\left(\frac{4}{5}\right)^n$
$\log_{4/5}\left(\frac{1}{25}\right) > n$
$\Rightarrow n > 14.4$

Method 2:
$\log\left(\frac{1}{25}\right) > \log\left(\frac{4}{5}\right)^n$
$\log\left(\frac{1}{25}\right) > n\log\left(\frac{4}{5}\right)$
$n > \frac{\log(1/25)}{\log(4/5)}$
> 14.4

$\underline{n = 15}$

(b) $y = x^3 - 8x^2 + 20x$

a) $\frac{dy}{dx} = 3x^2 - 16x + 20$

At stationary points, $\frac{dy}{dx} = 0$:

$$3x^2 - 16x + 20 = 0$$
$$(3x - 10)(x - 2) = 0$$
$$x = \tfrac{10}{3} \text{ or } x = 2$$

If $x = \frac{10}{3}$, $y = \left(\frac{10}{3}\right)^3 - 8\left(\frac{10}{3}\right)^2 + 20\left(\frac{10}{3}\right) = \frac{400}{27}$ B $\underline{\left(\frac{10}{3}, \frac{400}{27}\right)}$

If $x = 2$, $y = 2^3 - 8 \times 2^2 + 20 \times 2 = 16$.   A $\underline{(2, 16)}$

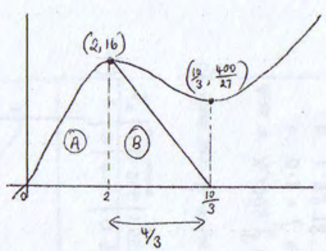
b) $\frac{d^2y}{dx^2} = 6x - 16$

When $x = 2$, $\frac{d^2y}{dx^2} = 6 \times 2 - 16 = -4 < 0 \Rightarrow$ A is a maximum.

c) $\int x^3 - 8x^2 + 20x \, dx$

$$= \frac{x^4}{4} - \frac{8x^3}{3} + 20\frac{x^2}{2} + C$$
$$= \underline{\underline{\frac{x^4}{4} - \frac{8x^3}{3} + 10x^2 + C}}$$

d)



Area A $= \int_0^2 x^3 - 8x^2 + 20x \, dx$

$$= \left[\frac{x^4}{4} - \frac{8x^3}{3} + 10x^2\right]_0^2$$
$$= \left[\left(\frac{2^4}{4} - \frac{8 \times 2^3}{3} + 10 \times 2^2\right) - (0)\right]$$
$$= \frac{68}{3}$$

Area B $= \frac{1}{2} \times \frac{4}{3} \times 16 = \frac{32}{3}$.

Total Area R $= \frac{68}{3} + \frac{32}{3} = \underline{\underline{\frac{100}{3}}}$