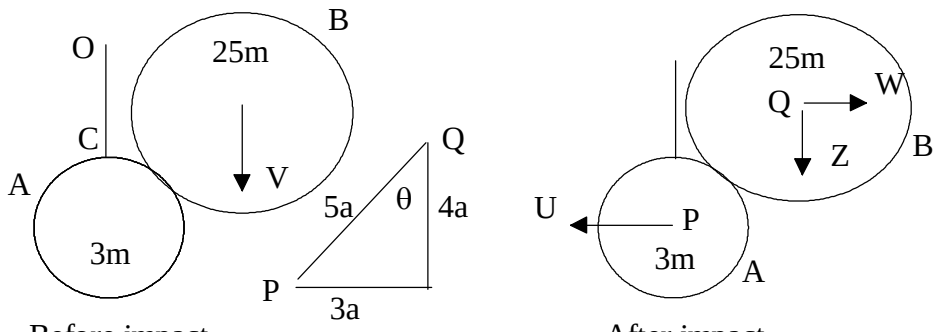
11..



Before impact After impact

By law of restitution, $eV\cos\theta = U\sin\theta - Z\cos\theta + W\sin\theta$ and $\tan\theta = \frac{3}{4} = e$ so $3V = 3U - 4Z + 3W$ (1)

By conservation of momentum horizontally, $3mU = 25mW$ (2)

By conservation of momentum in plane of contact, $W\cos\theta + 3Z\sin\theta = V\sin\theta \Rightarrow 4W + 3Z = 3V$ (3)

From (2), $W = \frac{3U}{25}$ and from (1) and (3) $4W + 3Z = 3U - 4Z + 3W \Rightarrow 7Z = 3U - W = \frac{72U}{25} \Rightarrow Z = \frac{72U}{175}$

substituting in (1) we have $3V = 3U - \frac{288U}{175} + \frac{9U}{25} = \frac{525-288+63}{175}U = \frac{300}{175}U = \frac{12}{7}U \Rightarrow V = \frac{4}{7}U$

i.e. $U = \frac{7}{4}V$

In second case let speed of A be X, we must now consider angular momentum about O

M.I. of A about O is $\frac{2}{5}(3m)(2a)^2 + 3m(5a)^2 = \frac{24}{5}ma^2 + 75ma^2 = \frac{399}{5}ma^2$

Equations (1) and (3) still hold with U replaced by X

i.e. $3V = 3X - 4Z + 3W$ and $4W + 3Z = 3V$ so eliminating Z we have $21V = 9X + 25W$

hence $W = \frac{21V-9X}{25}$ so $3Z = 3V - 4W = 3V - \frac{84V-36X}{25} = \frac{36X-9V}{25}$ so $Z = \frac{12X-3V}{25}$

by conservation of angular momentum about O

$25m \times 3aV = 25m \times 3aZ - 25m \times aW + \frac{399}{5}\boldsymbol{m}a^2\left(\frac{X}{5a}\right)$ i.e. $\frac{399}{25}X = 75V - 75Z + 25W$

hence $\frac{399}{25}X = 75V - 36X + 9V + 21V - 9X = 105V - 45X$

$\Rightarrow (399 + 45 \times 25)X = 105 \times 25V \Rightarrow X = \frac{105\times25}{1524}V = \frac{35\times25}{508} \times \frac{4}{7}U = \frac{125}{127}U$ Q.E.D.

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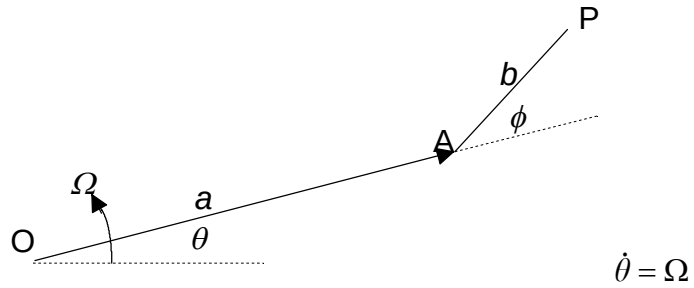
12.



Acceleration of A perpendicular to AP is $a\Omega^2 \sin\phi$
Acceleration of P perpendicular to AP and relartive to A is $b\ddot{\phi}$
So acceleration of P perpendicular to AP is $a\Omega^2 \sin\phi + b\ddot{\phi}$
Since pivot at A is smooth there is no force perpendicular to AP
so $a\Omega^2 \sin\phi + b\ddot{\phi} = 0 \Rightarrow \ddot{\phi} = -\frac{a}{b}\Omega^2 \sin\phi \Rightarrow 2\ddot{\phi}\dot{\phi} = -2\left(\frac{a}{b}\Omega^2 \sin\phi\right)\dot{\phi}$
integrating we then have $\left(\frac{d\phi}{dt}\right)^2 = \frac{2a}{b}\Omega^2 \cos\phi + C$
Initially, $\frac{d\phi}{dt} = \frac{1}{b}\left\{\left(a + 2\sqrt{ab}\right)\Omega - a\Omega\right\} = 2\sqrt{\frac{a}{b}}\,\Omega > 0$
Hence, $C = \frac{2a}{b}\Omega^2$ and so $\frac{d\phi}{dt} = \sqrt{\frac{2a}{b}\Omega^2(\cos\phi + 1)} = 2\sqrt{\frac{a}{b}}\cos\left(\frac{1}{2}\phi\right)$
Integrating $\int \sec\left(\frac{1}{2}\phi\right)d\phi = \int 2\sqrt{\frac{a}{b}}\,dt \Rightarrow \frac{1}{2}\ln\left(\sec\left(\frac{1}{2}\phi\right) + \tan\left(\frac{1}{2}\phi\right)\right) = 2\sqrt{\frac{a}{b}}\,t + C$
$\phi = 0$ when $t = 0$ so $C = 0$ and $\sec\left(\frac{1}{2}\phi\right) + \tan\left(\frac{1}{2}\phi\right) = e^{4\sqrt{\frac{a}{b}}\,t}$
ϕ can never be equal to π so AP cannot cross OP .

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13. Taking axes along and perpendicular to OA and using complex coordinates
Let holes in table be at positions defined by $a, a\omega, a\omega^2, a\omega^3, a\omega^4$ where $\omega^5 = 1$
if P is moved to position z then lengths of strings above table are $(z - aw^r)$ for $r = 0, 1, 2, 3, 4$
These lengths are also the extensions of the string so total force on particle P is
$\sum_{r=0}^{4} -\lambda(z - a\omega^r) = -\frac{5\lambda z}{a}$ since $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$
hence, $m\frac{d^2z}{dt^2} = -\frac{5\lambda z}{a} \Rightarrow \frac{d^2z}{dt^2} + \frac{5\lambda z}{am} = 0$ and motion is simple harmonic with period $2\pi\sqrt{\frac{am}{5\lambda}}$
If now the string PAO has modulus $k\lambda$ then total force from other four strings is now
$\sum_{r=1}^{4} -\lambda(z - a\omega^r) = -4\lambda z + a\lambda$ in direction PO
so we now have $m\frac{d^2z}{dt^2} = \frac{-4\lambda z + a\lambda - k\lambda(z-1)}{a} = \frac{-(4+k)\lambda z + (a+k)\lambda}{a}$
$\Rightarrow \frac{d^2z}{dt^2} + \frac{(4+k)\lambda z}{am} = \frac{(a+k)\lambda}{a}$
By symmetry the particle moves in a straight line with SHM with period $2\pi\sqrt{\frac{am}{(4+k)\lambda}}$
If this $= \frac{T}{2}$ then $4\pi\sqrt{\frac{am}{(4+k)\lambda}} = 2\pi\sqrt{\frac{am}{5\lambda}} \Rightarrow \frac{4am}{4+k} = \frac{am}{5} \Rightarrow 4 + k = 20 \Rightarrow k = 16$

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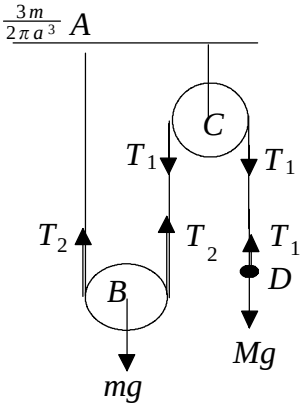
14. Consider a thin ring of radius x and width δx

the mass of this ring will be $2\pi x\rho\delta x$ where ρ is the density, but $\rho \propto x$, $= kx$ say

so mass $= 2k\pi x^2\delta x$ and total mass of disc, $m = \int_0^a 2k\pi x^2 \mathrm{d}x = \frac{2}{3}k\pi a^3 \Rightarrow k = \frac{3m}{2\pi a^3}$



MI of ring about perpendicular axis through centre $= 2\pi x^3\rho\delta x$

so MI of disc $= \int_0^a 2k\pi x^4 \mathrm{d}x = \frac{2}{5}k\pi a^5 = \frac{2\pi a^5}{5} \times \frac{3m}{2\pi a^3} = \frac{3}{5}ma^2$ as required.

Let acceleration of D be f then by Newton's second law

$Mg - T_1 = Mf$, $(T_1 - T_2)a = \frac{3}{5}ma^2\frac{f}{a}$ and $2T_2 - mg = \frac{1}{2}mf$

so $a(Mg - T_1) + (T_1 - T_2)a + \frac{a}{2}(2T_2 - mg) = aMf + \frac{3}{5}ma^2\frac{f}{a} + \frac{1}{4}maf$

$\Rightarrow aMg - \frac{a}{2}mg = \left(M + \frac{3}{5}m + \frac{1}{4}m\right)af = \left(M + \frac{17}{20}m\right)af$

$\Rightarrow f = \frac{(20M-10m)g}{20M+17m} > 0$ since $m < 2M$

velocity of D after time t is thus $\frac{(20M-10m)gt}{20M+17m}$

so impulse on D when suddenly brought to rest $= \frac{M(20M-10m)gt}{20M+17m}$

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15. (i) $\Pr(n$ independently selected values of X all $> k) = \left(\frac{c-k}{2c}\right)^n$

(ii) $\Pr(n$ independently selected values of X all $< k) = \left(\frac{k+c}{2c}\right)^n$

$\Pr(\text{median lies in interval of width } \delta) = \frac{\delta}{2c}$

Let median value be z then we must have n values less than z and n values greater than z

hence, $\Pr(\text{median} = z) = \binom{2n+1}{n}\left(\frac{z+c}{2c}\right)^n \times \binom{n+1}{n}\left(\frac{c-z}{2c}\right)^n \times \frac{\delta z}{2c}$

$$= \frac{(2n+1)!}{n!(n+1)!} \times \frac{(n+1)!}{n!1!} \times \left(\frac{z+c}{2c}\right)^n\left(\frac{c-z}{2c}\right)^n\frac{\delta z}{2c} = \frac{(2n+1)!}{(n!)^2} \cdot \frac{(c^2-z^2)\delta z}{(2c)^{2n+1}}$$

i.e. p.d.f. of $Z = \frac{(2n+1)!(c^2-z^2)}{(n!)^2(2c)^{2n+1}}$ as required

hence, $\int_{-c}^{c} \frac{(2n+1)!(c^2-z^2)}{(n!)^2(2c)^{2n+1}}\mathrm{d}z = 1 \Rightarrow \int_{-c}^{c}(c^2 - z^2)\mathrm{d}z = \frac{(n!)^2(2c)^{2n+1}}{(2n+1)!}$

$$\mathbf{E}(Z) = \int_{-c}^{c} \frac{(2n+1)!(c^2-z^2)z}{(n!)^2(2c)^{2n+1}}\mathrm{d}z = \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}\int_{-c}^{c}(c^2 - z^2)z\mathrm{d}z = \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}\left[c^2z - \frac{1}{3}z^3\right]_{-c}^{c}$$

$$= \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}\left(\frac{2}{3}c^3 + \frac{2}{3}c^3\right) = \frac{4(2n+1)!c^3}{(n!)^2(2c)^{2n+1}} = \frac{(2n+1)!c}{(n!)^2(2c)^{2n-1}}$$

$$\mathbf{E}(Z^2) = \int_{-c}^{c} \frac{(2n+1)!(c^2-z^2)z^2}{(n!)^2(2c)^{2n+1}}\mathrm{d}z = \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}\left[\frac{1}{3}c^2z^3 - \frac{1}{5}z^5\right]_{-c}^{c} = \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}} \times \frac{4c^5}{15}$$

so $\mathrm{Var}(Z) = \frac{(2n+1)!c^3}{(n!)^2(2c)^{2n-1}} - \left(\frac{(2n+1)!c}{(n!)^2(2c)^{2n-1}}\right)^2$

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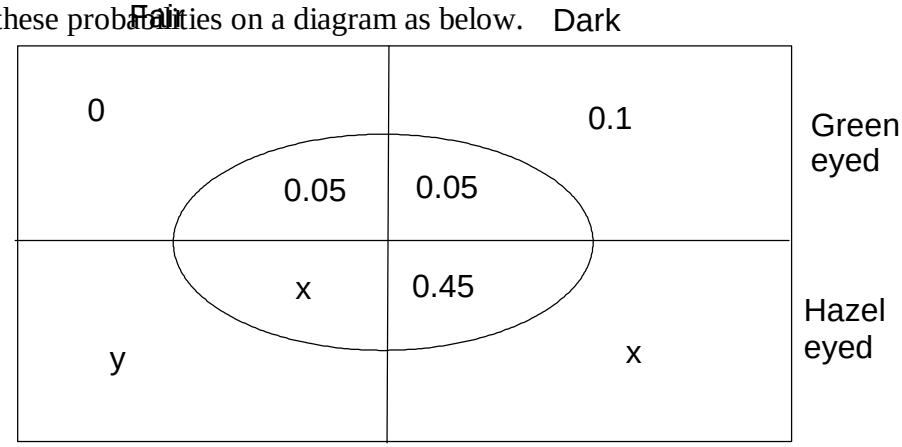
16. With obvious notation (i) $\mathrm{P}(F) = 0.25,\ \mathrm{P}(D) = 0.75,$ (ii) $\mathrm{P}(G) = 0.2,\ \mathrm{P}(H) = 0.8$
(iv) $\mathrm{P}(N \cap G \cap F) = 0,$ (v) $\mathrm{P}(G \cap N) = \mathrm{P}(G \cap B) = 0.1,$ (vi) $\mathrm{P}(G \cap B \cap F) = \mathrm{P}(G \cap B \cap D) = 0.05$
(vii) $\mathrm{P}(B \cap D) = 0.5,$ (viii) $\mathrm{P}(H \cap F \cap B) = \mathrm{P}(H \cap D \cap N)$
We represent these probabilities on a diagram as below.



Broad-headed inside ellipse, Narrow headed outside ellipse

$x + y + 0.05 = 0.25 \Rightarrow x + y = 0.2$ and $x + 0.6 = 0.75 \Rightarrow x = 0.15$ so $y = 0.05$
check. $\mathrm{P}(H) = 2x + y + 0.45 = 0.8$
so $\mathrm{P}(N) = 0 + 0.1 + x + y = 0.3$ i.e. we believe 30% are narrow headed

P(6 Ruritanians are all broad-headed) $= (0.7)^6 = 0.118$
This is not small enough for us to discount the given model but would give us cause for concern.

Number of narrow heads in samples of 200 $\sim \mathrm{B}(200, 0.3) \approx \mathrm{N}(60, 42)$
$\mathrm{P}(50 \text{ or fewer}) = \mathrm{P}\left(z < \frac{50.5-60}{\sqrt{42}}\right) = \mathrm{P}(z < -1.466) \approx 0.07$
This is reasonable evidenc that the model is not a good one.

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