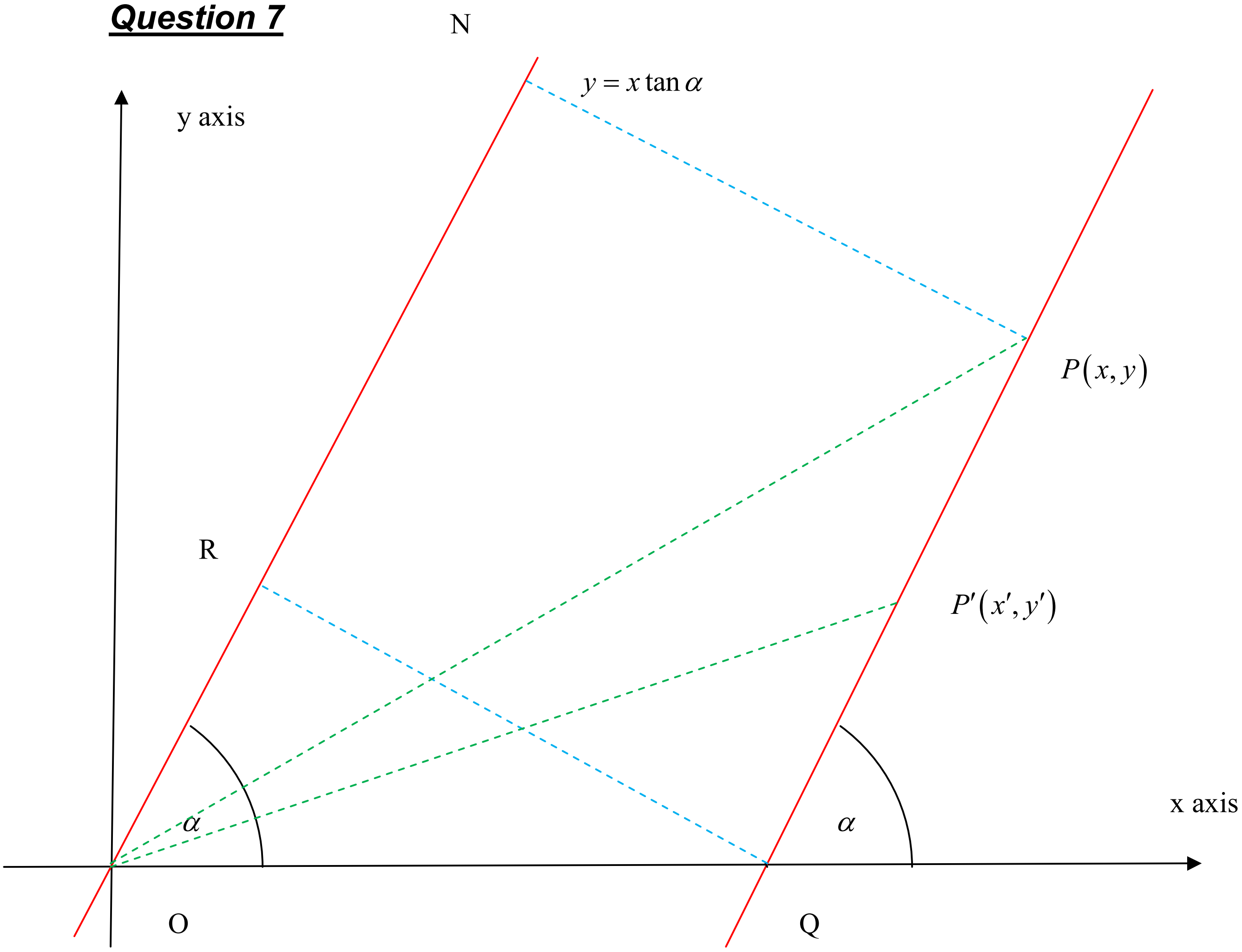
Question 7



Let PP' meet the x axis at Q.

Draw the line QR parallel to PN.

Q is the point $(x - PQ\cos\alpha, 0) = \left(x - \dfrac{y}{\sin\alpha}\cos\alpha, 0\right) = (x - y\cot\alpha, 0)$

Hence
$$\begin{aligned} x' = x - PP'\cos\alpha &= x - kPN\cos\alpha \\ &= x - kQR\cos\alpha \\ &= x - kOQ\sin\alpha\cos\alpha \\ &= x - k(x - y\cot\alpha)\sin\alpha\cos\alpha \\ &= x(1 - k\sin\alpha\cos\alpha) + yk\cos^2\alpha \end{aligned}$$

Similarly $y' = y - PP'\sin\alpha = y - k\left(x - y\cot\alpha\right)\sin^2\alpha$

$$= y\left(1 + k\sin\alpha\cos\alpha\right) - xk\sin^2\alpha$$

Hence $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 - k\sin\alpha\cos\alpha & k\cos^2\alpha \\ -k\sin^2\alpha & 1 + k\sin\alpha\cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Consider $\begin{pmatrix} p & q \\ r & t \end{pmatrix}\begin{pmatrix} 1 - k\sin\alpha\cos\alpha & k\cos^2\alpha \\ -k\sin^2\alpha & 1 + k\sin\alpha\cos\alpha \end{pmatrix}$

$$= \begin{pmatrix} p\left(1 - k\sin\alpha\cos\alpha\right) - qk\sin^2\alpha & pk\cos^2\alpha + q\left(1 + k\sin\alpha\cos\alpha\right) \\ r\left(1 - k\sin\alpha\cos\alpha\right) - tk\sin^2\alpha & rk\cos^2\alpha + t\left(1 + k\sin\alpha\cos\alpha\right) \end{pmatrix}$$

And $\begin{pmatrix} 1 - k\sin\alpha\cos\alpha & k\cos^2\alpha \\ -k\sin^2\alpha & 1 + k\sin\alpha\cos\alpha \end{pmatrix}\begin{pmatrix} p & q \\ r & t \end{pmatrix}$

$$= \begin{pmatrix} p\left(1 - k\sin\alpha\cos\alpha\right) + rk\cos^2\alpha & q\left(1 - k\sin\alpha\cos\alpha\right) + tk\cos^2\alpha \\ r\left(1 + k\sin\alpha\cos\alpha\right) - pk\sin^2\alpha & t\left(1 + k\sin\alpha\cos\alpha\right) - qk\sin^2\alpha \end{pmatrix}$$

The top elements on the leading diagonal are equal if and only if :-

$$p\left(1 - k\sin\alpha\cos\alpha\right) - qk\sin^2\alpha = p\left(1 - k\sin\alpha\cos\alpha\right) + rk\cos^2\alpha$$
$$\Leftrightarrow -q\sin^2\alpha = r\cos^2\alpha$$
$$\Leftrightarrow q\tan\alpha = -r\cot\alpha$$

This also ensures that the bottom elements of the leading diagonal are equal!

The top elements of the trailing diagonal are equal if and only if:-

$$pk\cos^2\alpha + q\left(1 + k\sin\alpha\cos\alpha\right) = q\left(1 - k\sin\alpha\cos\alpha\right) + tk\cos^2\alpha$$
$$\Leftrightarrow \left(p - t\right)k\cos^2\alpha = -2qk\sin\alpha\cos\alpha$$
$$\Leftrightarrow p - t = -2q\tan\alpha$$

Therefore the three elements compared so far are equal if and only if:-

$$t - p = 2q\tan\alpha = -2r\cot\alpha$$

But this also implies that the fourth element (i.e. the bottom of leading diagonal pair) will be equal because:-

$$r\left(1-k\sin\alpha\cos\alpha\right)-tk\sin^2\alpha = r\left(1+k\sin\alpha\cos\alpha\right)-pk\sin^2\alpha$$

$$\Leftrightarrow 2rk\sin\alpha\cos\alpha = \left(p-t\right)k\sin^2\alpha$$

$$\Leftrightarrow p-t = 2r\cot\alpha$$

Therefore $\begin{pmatrix} p & q \\ r & t \end{pmatrix}$ and $\begin{pmatrix} 1-k\sin\alpha\cos\alpha & k\cos^2\alpha \\ -k\sin^2\alpha & 1+k\sin\alpha\cos\alpha \end{pmatrix}$ commute if and only if

$$t-p = 2q\tan\alpha = -2r\cot\alpha$$
