Imperial College London

Department of Materials

MSE 201: Mathematics

Fourier Methods

Dr Arash Mostofi

2010-11

Comments and corrections to a.mostofi@imperial.ac.uk

Lecture notes may be found on Blackboard (http://blackboard.ic.ac.uk)

Contents

Chapter 1

Fourier Methods

1.1 Learning Outcomes

To understand the concept of orthogonality of functions, with sine and cosine as particularly important examples. To understand the concept of periodic functions. To be able to represent periodic functions in terms of a Fourier series of sine and cosine functions. The case of discontinuous functions. Parseval's theorem for Fourier series and it's relation to power spectra. To understand the relationship between Fourier series and Fourier transforms. To know the definition and fundamental properties of the Fourier transform. The Parseval and convolution theorems for Fourier transforms. Application of Fourier methods to diffraction and differential equations.

1.2 Further Reading

It is important that you get used to reading material beyond the lecture notes. This has a number of benefits: it allows you to explore material that goes beyond the core concepts included here; it encourages you to develop your skills in assimilating academic literature; seeing a subject presented in a different ways by different authors can enhance your understanding.

1. Riley, Hobson & Bence, *Mathematical Methods for Physics and Engineering, 3rd Ed.*, Chapter 12: Fourier Series; Chapter 13: Integral Transforms, Section 1: Fourier Transforms.

2. Arfken & Weber, *Mathematical Methods for Physicists, 4th Ed.*, Chapter 14: Fourier Series; Chapter 15: Integral Transforms, Sections 1 to 5.

3. Boas, *Mathematical Methods in the Physical Sciences, 2nd Ed.*, Chapter 7: Fourier Series; Chapter 15: Integral Transforms, Sections 4, 5 and 7.

1.3 Introduction

You are familiar with the concept that any vector $\mathbf{v}$ may be represented as a linear combination of a set of suitable basis vectors, e.g., the orthonormal Cartesian unit vectors in three dimensions: $\{\hat{\mathbf{e}}_i\}$:

$$\mathbf{v} = \sum_{i=1}^{3} a_i \hat{\mathbf{e}}_i.$$

The same is true of functions. Functions may be thought of as generalised, infinite dimensional vectors that live in an infinite-dimensional vector-space (also called a *Hilbert space*[1]). If one can find a suitable basis of functions $\{B_i(x)\}$ for a given Hilbert space, then any function $f(x)$ in that space may be expressed as a linear combination:

$$f(x) = \sum_i a_i B_i(x). \tag{1.1}$$

One important class of functions is that of periodic functions, and a particularly useful set of basis functions for the Hilbert space of periodic functions is that of sines and cosines, also known as the Fourier[2] basis. If $\{B_i(x)\}$ is a Fourier basis, then an expansion such as that of Eq. 1.1 is called a *Fourier series*.

Fourier series (and transforms) are very important in science and engineering and have applications in signal processing and electronics, solving differential equations, image processing and image enhancement, diffraction and interferometry, and much more besides.

1.4 Orthogonality of Functions

1.4.1 Recap of Vectors

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal if their *inner product*[3] is zero: $\mathbf{a} \cdot \mathbf{b} = 0$. Expanding $\mathbf{a}$ and $\mathbf{b}$ in terms of orthonormal Cartesian basis vectors,

$$\mathbf{a} = \sum_{i=1}^{3} a_i \hat{\mathbf{e}}_i \qquad \mathbf{b} = \sum_{i=1}^{3} b_i \hat{\mathbf{e}}_i$$

[1] David Hilbert (1862-1943), German mathematician. Developed much of the mathematical foundation for both general relativity and quantum mechanics.

[2] Jean Baptiste Joseph Fourier (1768–1830), French mathematician and physicist. Fourier also discovered the greenhouse effect.

[3] Also known as *scalar product* or *dot product*.

and using the usual orthonormality relations $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$,[4] we find that[5]

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \sum_{i,j=1}^{3} a_i^* b_j \, \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \\ &= \sum_{i,j=1}^{3} a_i^* b_j \delta_{ij} \\ &= \sum_{i=1}^{3} a_i^* b_i. \end{aligned}$$

So, the condition for $\mathbf{a}$ and $\mathbf{b}$ to be mutually orthogonal is

$$\boxed{\mathbf{a} \cdot \mathbf{b} = \textstyle\sum_{i=1}^{3} a_i^* b_i = 0}$$

This was in three dimensions and the extension to N-dimensional vectors is trivial:

$$\boxed{\mathbf{a} \cdot \mathbf{b} = \textstyle\sum_{i=1}^{N} a_i^* b_i = 0}$$

1.4.2 Extension to Functions

Since functions may be thought of as infinite dimensional vectors, the concept of orthogonality can be applied to functions as well. The inner product of two functions $f(x)$ and $g(x)$ is defined as[6]

$$\boxed{\langle f | g \rangle \equiv \int_a^b f^*(x) g(x) \, \mathrm{d}x}$$

Two functions $f(x)$ and $g(x)$ are said to be orthogonal on the interval $[a, b]$ if their inner product is zero: $\langle f | g \rangle = 0$.

[4]Recall the Krönecker delta is the suffix notation analogue of the identity matrix, in other words, $\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$

[5]The asterisk $*$ denotes the complex conjugate. You don't need to worry about it when dealing with vectors whose components a_i are real, but this is not always the case.

[6]The complex conjugate ensures that $\langle f | f \rangle \geq 0$ for complex valued functions.

Exercise: Sketch the polynomials given by

$$P_1(x) = x \qquad P_2(x) = \frac{3x^2-1}{2} \qquad P_3(x) = \frac{5x^3-3x}{2}$$

and show that they are mutually orthogonal on the interval $[-1, 1]$.

1.4.3 The Fourier Basis

Consider the set of functions $s_n(x) = \sin(\frac{n\pi x}{L})$, $n \in \{1, 2, \ldots\}$. They have a common period of $2L$, i.e., $s_n(x + 2L) = s_n(x)$.

Exercise: Sketch s_1, s_2 and s_3 in the interval $x \in [-L, L]$.

The functions $\{s_n(x)\}$ form a mutually orthogonal set over any interval of length $2L$, i.e., $\langle s_m|s_n\rangle \propto \delta_{mn}$.

Proof: Consider the symmetric interval $[-L, L]$.

$$\begin{aligned}
\langle s_m|s_n\rangle &= \int_{-L}^{L} s_m^*(x)s_n(x)\,\mathrm{d}x \\
&= \int_{-L}^{L} \sin(\tfrac{m\pi x}{L})\sin(\tfrac{n\pi x}{L})\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-L}^{L} \left[\cos(\tfrac{(m-n)\pi x}{L}) - \cos(\tfrac{(m+n)\pi x}{L})\right]\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin(\tfrac{(m-n)\pi x}{L}) - \frac{L}{\pi(m+n)}\sin(\tfrac{(m+n)\pi x}{L})\right]_{-L}^{L} && m \neq n \\
&= 0 && \text{since } \sin p\pi = 0 \text{ for any integer } p
\end{aligned}$$

For the case of $m = n$, we have

$$\begin{aligned}
\langle s_n|s_n\rangle &= \int_{-L}^{L} s_n^*(x)s_n(x)\,\mathrm{d}x \\
&= \int_{-L}^{L} \sin^2(\tfrac{n\pi x}{L})\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-L}^{L} \left[1 - \cos(\tfrac{2n\pi x}{L})\right]\mathrm{d}x \\
&= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin(\tfrac{2n\pi x}{L})\right]_{-L}^{L} \\
&= \frac{1}{2}\cdot 2L = L
\end{aligned}$$

Therefore, we have

$$\boxed{\int_{-L}^{L} \sin(\tfrac{m\pi x}{L}) \sin(\tfrac{n\pi x}{L})\, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \end{cases}} \tag{1.2}$$

In a similar way, the set of functions $c_n(x) = \cos(\frac{n\pi x}{L})$, $n \in \{0, 1, \ldots\}$, can be shown to be mutually orthogonal on any interval of length $2L$, e.g.,

$$\boxed{\int_{-L}^{L} \cos(\tfrac{m\pi x}{L}) \cos(\tfrac{n\pi x}{L})\, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \\ 2L & \text{if } m = n = 0 \end{cases}} \tag{1.3}$$

It is also easy to show that $c_n(x)$ and $s_n(x)$ are orthogonal to eachother on any interval of length $2L$:

$$\boxed{\int_{-L}^{L} \cos(\tfrac{m\pi x}{L}) \sin(\tfrac{n\pi x}{L})\, \mathrm{d}x = 0 \quad \forall\ m, n} \tag{1.4}$$

The orthogonality relations of equations 1.2, 1.3 and 1.4 form the fundamental basis of Fourier theory[7]. They may be written more concisely as

$$\boxed{\begin{array}{l} \langle s_m | s_n \rangle = \langle c_m | c_n \rangle = L\delta_{mn} \quad m, n \neq 0 \\[2ex] \langle c_0 | c_0 \rangle = 2L \\[2ex] \langle c_m | s_n \rangle = 0 \quad \forall\ m, n \end{array}}$$

The set of functions comprising $s_n(x)$ ($n \in \{1, 2, \ldots\}$) and $c_n(x)$ ($n \in \{0, 1, \ldots\}$) form what is known as the Fourier basis, an orthogonal basis for the Hilbert space of functions that are periodic (in this case with period $2L$).

This means that, in the same way that any vector may be expressed as a linear combination of basis vectors, any periodic function $f(x) = f(x + 2L)$ may be represented as a linear combination of sines and cosines with the same periodicity,

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(\tfrac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi x}{L}).$$

The expression above is known as a Fourier series, and Fourier analysis is all about finding the expansion coefficients $\{a_n\}$ and $\{b_n\}$.

[7]Similar orthogonality relations exist for the complex Fourier basis functions $\mathrm{e}^{in\pi x/L}$, $n \in \{-\infty, \ldots, \infty\}$.

1.5 Fourier Series

A periodic function may be written as a Fourier series, i.e., as a sum of sinusoidal *harmonics* with the same periodicity.

Consider a triangular wave $f(t)$, periodic on the interval $t \in [0, T]$

$$f(t) = \begin{cases} 4t/T & t \in [0, T/4] \\ 2(1 - 2t/T) & t \in [T/4, 3T/4] \\ 4(t/T - 1) & t \in [3T/4, T] \end{cases}$$

Its Fourier series is given by[8]

$$f(t) = \sum_{n=1}^{\infty} \frac{8 \cdot (-1)^{n+1}}{(2n-1)^2 \pi^2} \sin(\tfrac{2(2n-1)\pi t}{T}). \tag{1.5}$$

Writing out the first few terms explicitly:

$$f(t) = \frac{8}{\pi^2} \left(\sin(\tfrac{2\pi t}{T}) - \frac{1}{9} \sin(\tfrac{6\pi t}{T}) + \frac{1}{25} \sin(\tfrac{10\pi t}{T}) - \frac{1}{49} \sin(\tfrac{14\pi t}{T}) + \dots \right)$$

Figure 1.1 shows that by including more and more terms in the Fourier series, successively better approximations to $f(t)$ are obtained.

The coefficients of the expansion in Eqn. 1.5 are known as the *Fourier coefficients* and can be found by orthogonality.

1.5.1 Determining Fourier Coefficients

Recall that the components of a vector can be determined by orthogonality of the basis vectors. For example, let

$$\mathbf{v} = v_1 \hat{\mathbf{e}}_1 + v_2 \hat{\mathbf{e}}_2 + v_3 \hat{\mathbf{e}}_3.$$

Then, taking the inner (i.e., scalar) product of both sides of the above expression with $\hat{\mathbf{e}}_1$ and using the orthonormality of the $\{\hat{\mathbf{e}}_i\}$ we obtain

$$\begin{aligned} \mathbf{v} \cdot \hat{\mathbf{e}}_1 &= v_1 \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 + v_2 \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 + v_3 \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 \\ &= v_1. \end{aligned}$$

The general result is that $v_i = \mathbf{v} \cdot \hat{\mathbf{e}}_i$.

[8] We shall see in due course how to calculate Fourier series explicitly.

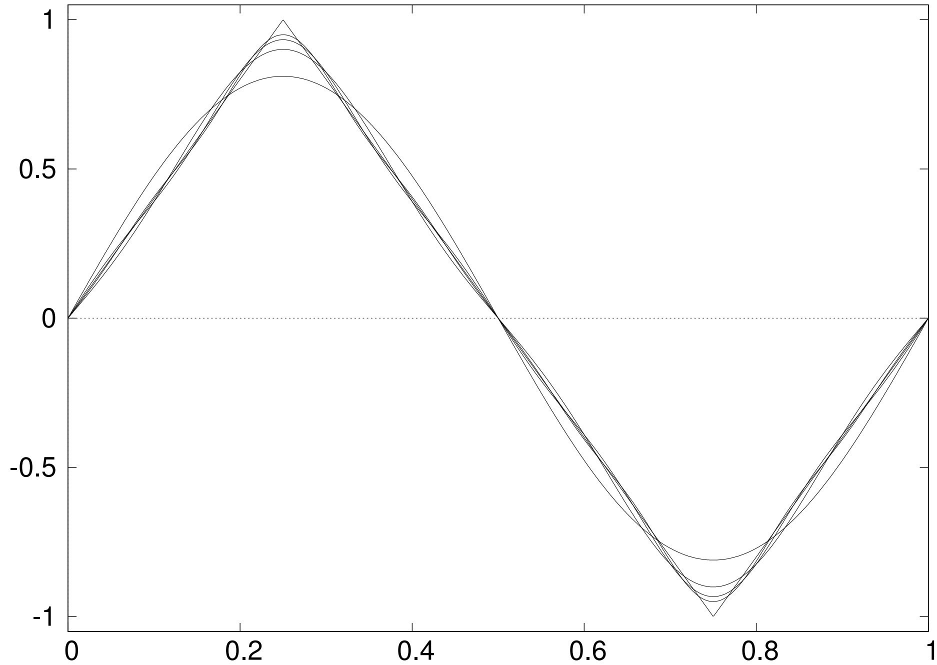


Figure 1.1: Triangular wave $f(t)$ (with $T = 1$) along with it's Fourier series representations consisting of 1, 2, 3, and 4 terms.

Now consider $f(x) = f(x + 2L)$, a function that is periodic with a period $2L$. It may be expressed as a Fourier series in terms of sines and cosines with the same periodicity, namely

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n \cos(\tfrac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi x}{L}) \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi x}{L}), \end{aligned} \tag{1.6}$$

where the $n = 0$ term has been explicitly separated from the first summation. The constants $\{a_n\}$ and $\{b_n\}$ are the Fourier coefficients and are analogous to the components of a vector. By analogy, the coefficients may be found by taking the inner product of $f(x)$ with the Fourier basis functions and using the orthogonality relations of Eqns. 1.2, 1.3 and 1.4:

$$\begin{aligned} \langle f|s_m\rangle &= a_0 \underbrace{\langle c_0|s_m\rangle}_{0} + \sum_{n=1}^{\infty} a_n \underbrace{\langle c_n|s_m\rangle}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\langle s_n|s_m\rangle}_{L\delta_{nm}} \\ &= Lb_m. \end{aligned}$$

Where we have used the shorthand notation $c_0(x) = 1$, $c_n(x) = \cos\frac{n\pi x}{L}$, and $s_n(x) = \sin\frac{n\pi x}{L}$. Similarly for a_m, $m \neq 0$:

$$\begin{aligned} \langle f|c_m\rangle &= a_0 \underbrace{\langle c_0|c_m\rangle}_{0} + \sum_{n=1}^{\infty} a_n \underbrace{\langle c_n|c_m\rangle}_{L\delta_{nm}} + \sum_{n=1}^{\infty} b_n \underbrace{\langle s_n|c_m\rangle}_{0} \\ &= La_m. \end{aligned}$$

And for a_0:

$$\begin{aligned}\langle f|c_0\rangle &= a_0\underbrace{\langle c_0|c_0\rangle}_{2L}+\sum_{n=1}^{\infty}a_n\underbrace{\langle c_n|c_0\rangle}_{0}+\sum_{n=1}^{\infty}b_n\underbrace{\langle s_n|c_0\rangle}_{0}\\ &= 2La_0.\end{aligned}$$

Therefore, in summary, the Fourier coefficients for the Fourier series of Eqn. 1.6 are given by[9]

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L}\langle f|c_0\rangle = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x \\ a_n &= \frac{1}{L}\langle f|c_n\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\tfrac{n\pi x}{L})\,\mathrm{d}x \quad n\geq 1 \\ b_n &= \frac{1}{L}\langle f|s_n\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\tfrac{n\pi x}{L})\,\mathrm{d}x \quad n\geq 1 \end{aligned}} \tag{1.7}$$

Exercise: Find the Fourier series for the periodic sawtooth function given by

$$f(x)=x, \quad x\in[-1,1], \quad f(x)=f(x+2)$$

1. Sketch $f(x)$.

2. Write down the general expression for the Fourier series:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(k_nx)+\sum_{n=1}^{\infty}b_n\sin(k_nx).$$

Q. What are the allowed values of k_n given the periodicity of $f(x)$?

A. Since $f(x+2)=f(x)$, then $\cos(k_n(x+2))=\cos(k_nx)$ which implies that $2k_n=2n\pi \Rightarrow k_n=n\pi$ (the result for $\sin(k_nx)$ is the same). Therefore

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(n\pi x)+\sum_{n=1}^{\infty}b_n\sin(n\pi x).$$

3. Either from first-principles, or using Eqns. 1.7, evaluate the coefficients a_0, $\{a_n\}$ and $\{b_n\}$:

The final result is

$$f(x)=\sum_{n=1}^{\infty}\frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x). \tag{1.8}$$

[9]Note that a_0 is the mean value of $f(x)$ over one period.

Writing out the first few terms explicitly:

$$f(x) = \frac{2}{\pi}\left[\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) - \frac{1}{4}\sin(4\pi x) + \ldots\right]$$

Figure 1.2 shows $f(x)$ and Fourier series approximations to $f(x)$ with successively more terms.

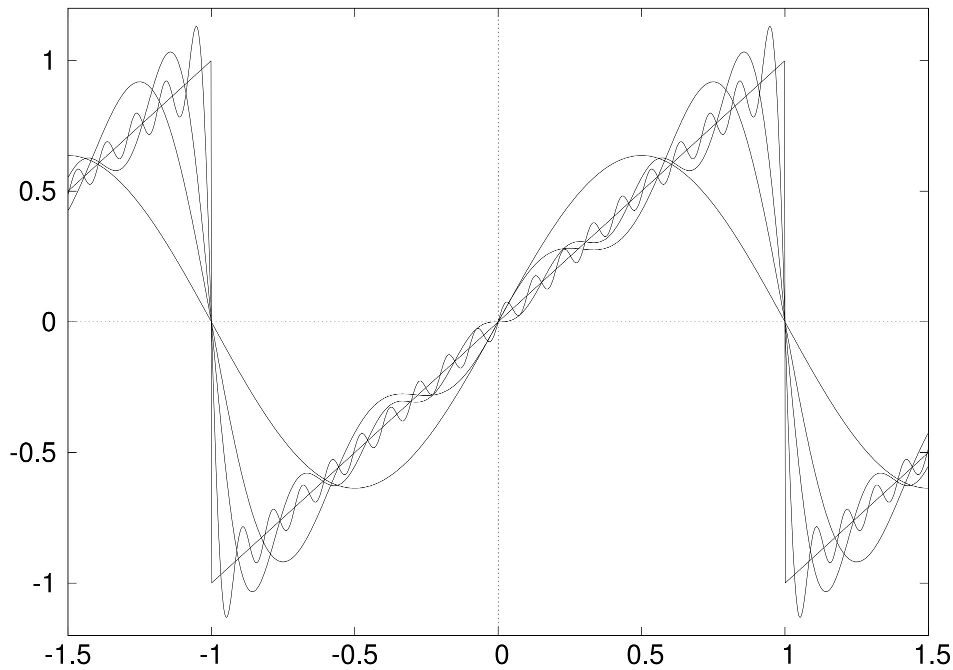


Figure 1.2: Sawtooth function $f(x)$ along with it's Fourier series representations consisting of 1, 3, 6, and 20 terms.

This example highlights a number of interesting features and properties of Fourier series. For example, why was $a_n = 0$ for all n? Why does the Fourier series for the sawtooth funtion converge more slowly than that for the triangular wave? What is going on with the apparent overshoot of the Fourier series for the sawtooth function at $x = \pm 1$?

1.5.2 Symmetry

A great deal of effort can be saved if one considers the symmetry of $f(x)$ in advance. Take the general Fourier series for a periodic function $f(x + 2L) = f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(\tfrac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi x}{L})$$

and note that $\cos(x) = \cos(-x)$ is an even function, while $\sin(x) = -\sin(-x)$ is an odd function. Then it follows that

- if $f(x)$ is odd, i.e., $f(x) = -f(-x)$, then only sine terms can be present in the Fourier series and $a_n = 0 \; \forall \; n$,
- and if $f(x)$ is even, i.e., $f(x) = f(-x)$, then only cosine terms can be present in the Fourier series and $b_n = 0 \; \forall \; n$.

1.6 Discontinuous Functions

Consider Fig. 1.2 again. The sawtooth function $f(x)$ is discontinuous at $x = \pm 1$. Notice how the value of the Fourier series at these points takes the arithmetic mean value of $f(x)$ on either side of the discontinuity. This is a general result: if $x = x_0$ is a point of discontinuity of $f(x)$, then the value of the Fourier series $F(x)$ at $x = x_0$ is

$$F(x_0) = \tfrac{1}{2}\left[f(x_0+) + f(x_0-)\right]$$

1.6.1 Convergence

Compare the Fourier series for the triangular wave (Eqn. 1.5 and Fig. 1.1) and the sawtooth function (Eqn. 1.8 and Fig. 1.2). What is noticeable is that for the former, the Fourier coefficients decrease as n^{-2}, while for the latter they decrease more slowly as n^{-1}. In other words, the Fourier series for the sawtooth function converges more slowly as the importance of successive terms does not decrease as fast. The difference is due to the fact that the sawtooth function is discontinuous, while the triangular wave is continuous and has a discontinuous first derivative.

In general if $f^{(k-1)}(x)$ is continuous, but $f^{(k)}(x)$ is not, then the Fourier coefficients will decrease as $n^{-(k+1)}$.

1.6.2 Gibbs Phenomenon

Take a closer look at Fig. 1.2. Let us denote the Fourier series truncated at the N^{th} term by $F_N(x)$:

$$F_N(x) = \sum_{n=0}^{N} a_n \cos(k_n x) + \sum_{n=1}^{N} \sin(k_n x).$$

What is clear is that as more terms of the Fourier series are included, a better representation of the function $f(x)$ is achieved in the sense that for a given value of x

$$|F_N(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

However, if we zoom in on the discontinuity in Fig. 1.2, as shown in Fig. 1.3, we see that there is a significant overshoot of $F_N(x)$. Furthermore, as we add more terms to the series, although the position of the overshoot moves further towards the discontinuity, it's magnitude remains

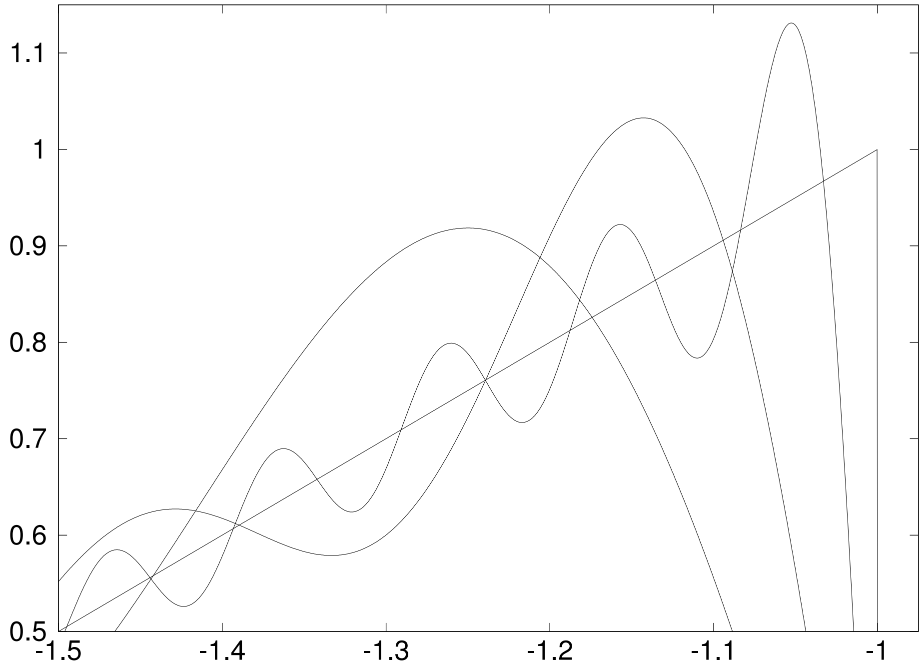


Figure 1.3: Sawtooth function $f(x)$ along with it's Fourier series representations consisting of 3, 6, and 20 terms.

finite. This is called the *Gibbs phenomenon* and it means that there is always an error in representing a discontinuous function[10]. By adding in more terms the size of the region over which there is an overshoot may be decreased, but the overshoot will always remain.

1.7 Half-Range Fourier Series

Consider the function

$$f(x) = 1 - x^2 \quad x \in [0, 1].$$

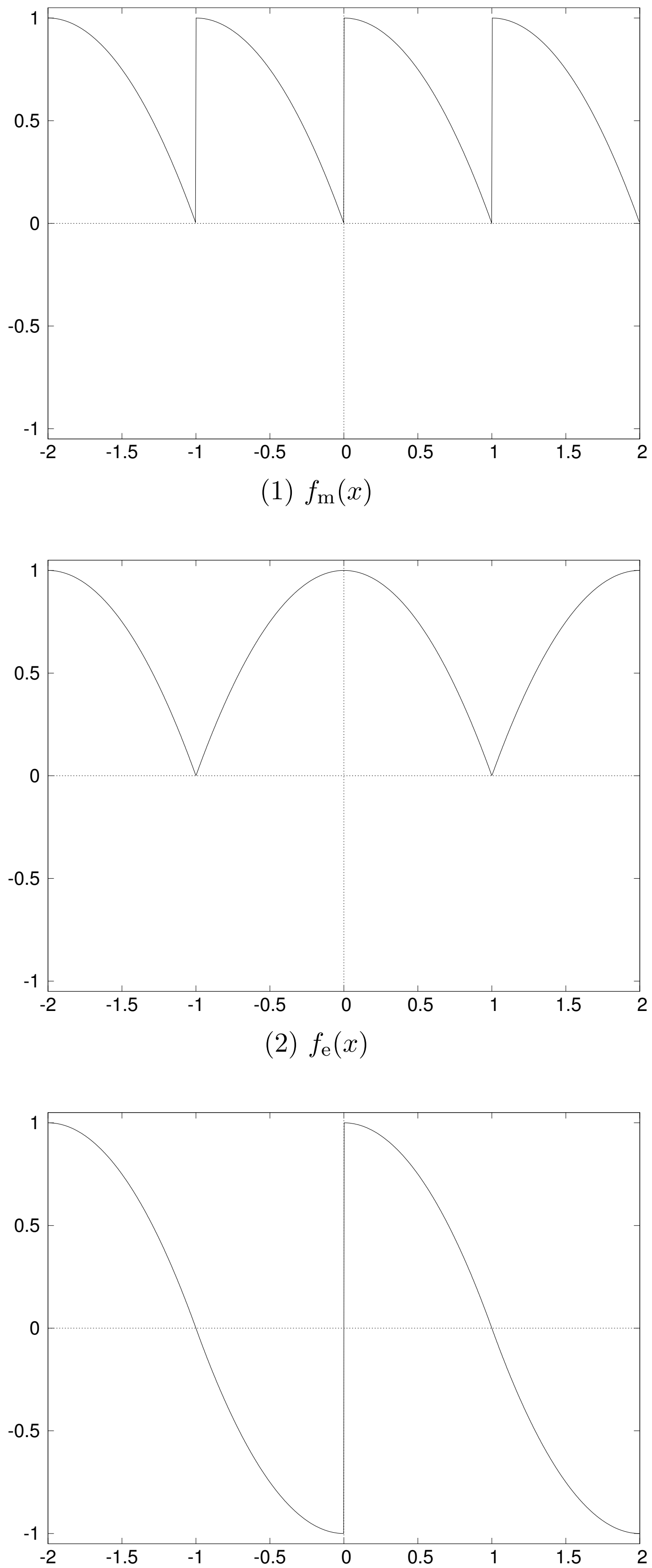
We have a number of choices for the periodic continuation of this function, $f_{\mathrm{m}}(x)$, $f_{\mathrm{e}}(x)$ and $f_{\mathrm{o}}(x)$, as plotted in Fig. 1.4.

In each case, the form of the Fourier series will be different.

1. $f_{\mathrm{m}}(x)$ has a period of 1, and is of mixed odd/even symmetry, therefore the Fourier coefficients of both sine and cosine terms will in general be non-zero and the series will be of the general form:

$$f_{\mathrm{m}}(x) = a_0 + \sum_{i=1}^{\infty} a_n \cos(2n\pi x) + \sum_{n=1}^{\infty} b_n \sin(2n\pi x)$$

[10]This phenomenon is not unique to Fourier series and is true of other basis function expansions.



(1) $f_{\mathrm{m}}(x)$

(2) $f_{\mathrm{e}}(x)$

(3) $f_{\mathrm{o}}(x)$

Figure 1.4: Three different periodic continuations of $f(x) = 1 - x^2$ for $x \in [0, 1]$.

The argument of the sine and cosines is chosen to reflect the periodicity of $f_{\rm m}(x)$, e.g., $\cos(2n\pi(x+1)) = \cos(2n\pi x)$.

2. $f_{\rm e}(x)$ has a period of 2, and is an *even* function of x, therefore the Fourier series will only contain even functions ($b_n = 0$):

$$f_{\rm e}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

Note that the argument of the cosine is chosen so that the Fourier series has the same periodicity as $f_{\rm e}(x)$: $\cos(n\pi(x+2)) = \cos(n\pi x)$.

3. $f_{\rm o}(x)$ has a period of 2, and is an *odd* function of x, therefore the Fourier series will only contain odd functions ($a_n = 0$):

$$f_{\rm o}(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

Again the argument of the sine is chosen so that the Fourier series has the same periodicity as $f_{\rm o}(x)$: $\sin(n\pi(x+2)) = \sin(n\pi x)$.

The expression for $f_{\rm m}(x)$ is called a *full-range* Fourier series, whereas those for $f_{\rm e}(x)$ and $f_{\rm o}(x)$ are called *half-range* Fourier series. Which is more "correct" to use depends on the circumstances and the particular problem that is being addressed.

Example: Find the half-range cosine Fourier series for $f_{\rm e}(x)$ above.

$$f_{\rm e}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

To find a_0, we should take the inner product of both sides of this expression with $c_0(x) = 1$, i.e., we should integrate the expression over a full period $[-1, 1]$ and use orthogonality. Because of symmetry of the integrands, however, the integral over $[-1, 0]$ is equal to that over $[0, 1]$, so actually we only need to integrate over $[0, 1]$:

$$\begin{aligned}
\int_{-1}^{1} f_{\rm e}(x)\cdot 1\,{\rm d}x &= a_0 \int_{-1}^{1} 1\,{\rm d}x + \sum_{n=1}^{\infty} a_n \int_{-1}^{1} \cos(n\pi x)\cdot 1\,{\rm d}x \\
2\int_{0}^{1} (1-x^2)\,{\rm d}x &= a_0\cdot 2\int_{0}^{1} {\rm d}x + \sum_{n=1}^{\infty} a_n\cdot 2\int_{0}^{1} \cos(n\pi x)\,{\rm d}x \\
\int_{0}^{1} (1-x^2)\,{\rm d}x &= a_0 \int_{0}^{1} {\rm d}x + \sum_{n=1}^{\infty} a_n \int_{0}^{1} \cos(n\pi x)\,{\rm d}x \\
\left[x - \frac{x^3}{3}\right]_0^1 &= a_0\,[x]_0^1 \qquad + \qquad 0 \\
\tfrac{2}{3} &= a_0
\end{aligned}$$

Similarly, to find a_m, we take the inner product with $c_m(x) = \cos(m\pi x)$. Again, we may exploit the symmetry of the integrands to only integrate over half of the period:

$$\begin{aligned}
\int_{-1}^{1} f_{\mathrm{e}}(x)\cos(m\pi x)\,\mathrm{d}x &= a_0\int_{-1}^{1}\cos(m\pi x)\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-1}^{1}\cos(n\pi x)\cos(m\pi x)\,\mathrm{d}x \\
\int_{0}^{1} (1-x^2)\cos(m\pi x)\,\mathrm{d}x &= a_0\int_{0}^{1}\cos(m\pi x)\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{0}^{1}\cos(n\pi x)\cos(m\pi x)\,\mathrm{d}x \\
\int_{0}^{1} (1-x^2)\cos(m\pi x)\,\mathrm{d}x &= 0 \qquad + \qquad \sum_{n=1}^{\infty} a_n \frac{1}{2}\delta_{nm} \\
&= \frac{a_m}{2}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{a_m}{2} &= \int_0^1 \cos(m\pi x)\,\mathrm{d}x - \int_0^1 x^2\cos(m\pi x)\,\mathrm{d}x \\
&= \underbrace{\left[\frac{\sin(m\pi x)}{m\pi}\right]_0^1}_{0} - \left\{ \underbrace{\left[x^2\frac{\sin(m\pi x)}{m\pi}\right]_0^1}_{0} - \int_0^1 2x\frac{\sin(m\pi x)}{m\pi}\,\mathrm{d}x \right\} \\
&= \int_0^1 2x\frac{\sin(m\pi x)}{m\pi}\,\mathrm{d}x \\
&= \left[-2x\frac{\cos(m\pi x)}{(m\pi)^2}\right]_0^1 + \frac{2}{(m\pi)^2}\underbrace{\int_0^1\cos(m\pi x)\,\mathrm{d}x}_{0} \\
\Rightarrow \quad \frac{a_m}{2} &= \frac{-2}{(m\pi)^2}\cos(m\pi) \\
\Rightarrow \quad a_m &= \frac{4}{(m\pi)^2}(-1)^{m+1}.
\end{aligned}$$

Hence

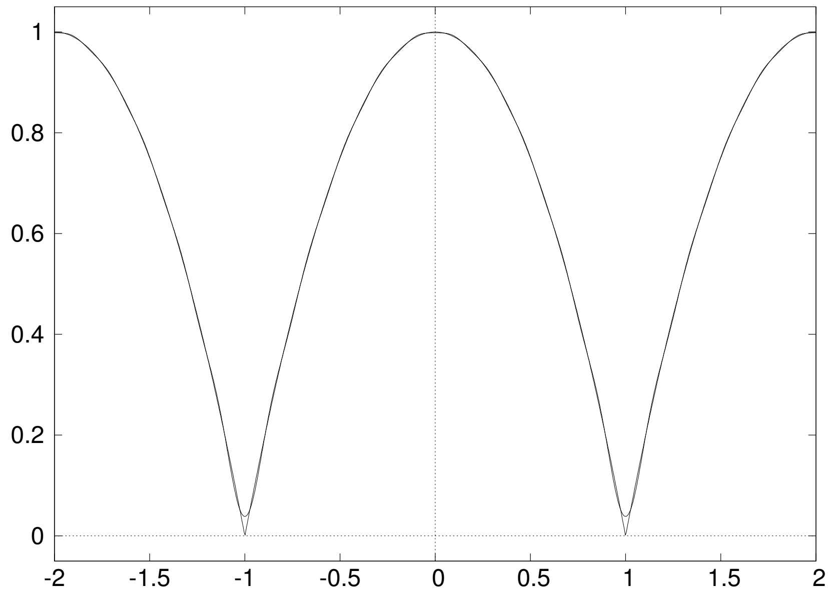
$$f_{\mathrm{e}}(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$$

Exercise: Find the half-range sine Fourier series for $f_{\mathrm{o}}(x)$ above (see question on examples sheet).

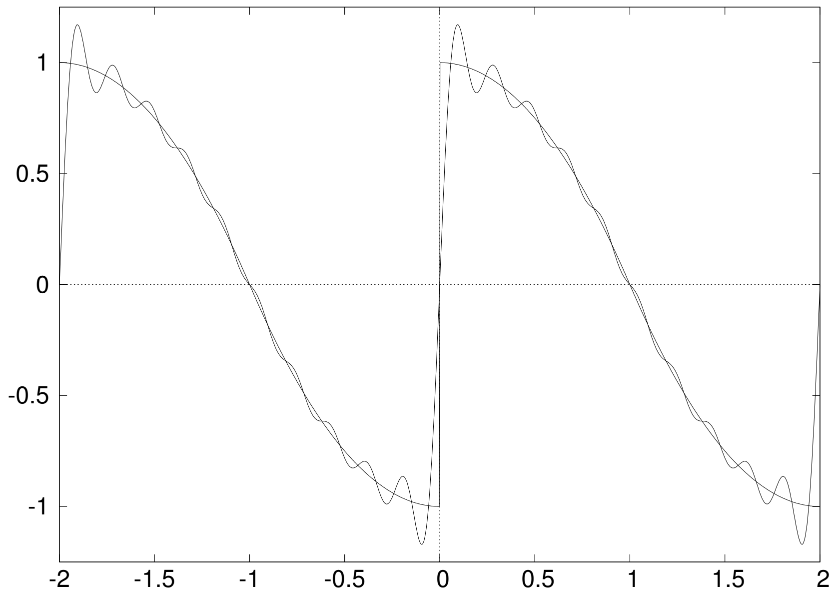
The result is

$$f_{\mathrm{o}}(x) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi} + \frac{4}{(n\pi)^3}\left[1-(-1)^n\right]\right)\sin(n\pi x).$$

Note the n^{-1} dependence of the Fourier coefficients, as compared to n^{-2} for the continuous function $f_{\mathrm{e}}(x)$. Fig. 1.5 shows the sum of the first ten terms of the Fourier series of $f_{\mathrm{e}}(x)$ and $f_{\mathrm{o}}(x)$.



(a) $f_e(x)$



(b) $f_o(x)$

Figure 1.5: Sum of the first ten terms of the Fourier series for (a) $f_e(x)$ and (b) $f_o(x)$.

1.8 Parseval's Theorem for Fourier Series

Consider a periodic function $f(x) = f(x + 2L)$. As before, its Fourier series may be written down generally as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi x}{L}).$$

We saw earlier that the mean value of $f(x)$ was simply a_0:

$$\frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x = a_0.$$

Now consider the mean value of $|f(x)|^2$:

$$\begin{aligned}\frac{1}{2L}\int_{-L}^{L} |f(x)|^2\,\mathrm{d}x &= \frac{1}{2L}\int_{-L}^{L}\left[a_0 + \sum_{m=1}^{\infty}\left\{a_m\cos(\tfrac{m\pi x}{L}) + b_m\sin(\tfrac{m\pi x}{L})\right\}\right] \\ &\qquad\times\left[a_0 + \sum_{n=1}^{\infty}\left\{a_n\cos(\tfrac{n\pi x}{L}) + b_n\sin(\tfrac{n\pi x}{L})\right\}\right] \\ \Rightarrow\quad \frac{1}{2L}\int_{-L}^{L} |f(x)|^2\,\mathrm{d}x &= a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right] \qquad (1.9)\end{aligned}$$

where all the cross terms have vanished by orthogonality. This is one form of *Parseval's theorem.*

In order to understand what it means, consider the n^{th} harmonic in the Fourier series of $f(x)$ is

$$h_n(x) = a_n\cos(\tfrac{n\pi x}{L}) + b_n\sin(\tfrac{n\pi x}{L}).$$

Using a suitable trigonometric identity, this may be expressed as

$$h_n(x) = \gamma_n\sin(\tfrac{n\pi x}{L} + \varphi_n),$$

where $\gamma_n = \sqrt{a_n^2 + b_n^2}$ and $\tan(\varphi_n) = \dfrac{a_n}{b_n}$. Comparing with Parseval's theorem, Eqn. 1.9, it can be seen that each harmonic in the Fourier series contributes independently to the mean square value of $f(x)$.[11]

In many applications in physics, for example where $f(x)$ represents a time-varying current in an electrical circuit, or an electromagnetic wave, the power density (energy per unit length transmitted per unit time) is proportional to the mean square value of $f(x)$, given by Eqn. 1.9, and the set of coefficients $a_0^2, \{\frac{1}{2}\gamma_n^2\}$ is known as the *power spectrum* of $f(x)$.

Exercise: Plot the power spectrum for $f_{\mathrm{e}}(x)$ and $f_{\mathrm{o}}(x)$ from the previous example.[12]

[11] This property is closely related to the *completeness* of the Fourier basis, which ensures that any function, periodic on a given interval, may be represented as a sum of sines and cosines.

[12] In other words, draw a bar chart of the power contributed by the harmonics as a function of n.

1.9 Fourier Transforms

1.9.1 Definition and Notation

Given $f(x)$ such that $\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty$, its *Fourier transform* $\tilde{f}(k)$ is defined as

$$\boxed{\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-ikx}\,\mathrm{d}x} \tag{1.10}$$

The *inverse Fourier transform* is defined as

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\,\mathrm{e}^{ikx}\,\mathrm{d}k} \tag{1.11}$$

A varied set of notation is used to represent Fourier transforms, the most common of which are given below.

$$\boxed{\tilde{f} \equiv \hat{f} \equiv F \equiv \mathfrak{F}[f] \equiv \mathrm{FT}[f] \qquad \text{and} \qquad f \equiv \mathfrak{F}^{-1}[\tilde{f}] \equiv \mathrm{FT}^{-1}[\tilde{f}]}$$

Fourier transforms are defined such that if one starts with $f(x)$ and performs a forward transform followed by an inverse transform on the result, then one ends up with $f(x)$ again,

$$f(x) \xrightarrow{\mathfrak{F}} \tilde{f}(k) \xrightarrow{\mathfrak{F}^{-1}} f(x)$$

which provides a definition of the *Dirac delta-function*, which we will come to in Section 1.11.

Conditions on $f(x)$

- Piecewise continuous
- Differentiable
- Absolutely integrable: $\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty$

1.9.2 Relation to Fourier Series

When we looked at Fourier series, we considered the Fourier basis used to expand a periodic function $f(x)$ as consisting of sine and cosine functions with the same periodicity as $f(x)$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(k_n x) + b_n \sin(k_n x)\right],$$

where k_n is chosen to satisfy the correct periodicity.

An entirely equivalent representation is provided by a basis of complex exponential functions (also known as *plane-waves*)

$$\mathrm{e}^{\pm i k_n x} = \cos(k_n x) \pm i \sin(k_n x).$$

Substituting this Euler[13] relation into the expression for $f(x)$ gives

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{i k_n x}$$

where the (complex) coefficient c_n is related to a_n and b_n (see question on examples sheet).

Consider the function to have periodicity of L, i.e., $f(x) = f(x+L)$. Then we require $k_n L = 2n\pi \Rightarrow k_n = \frac{2n\pi}{L}$. I.e., we have a sum of plane waves that have *wavevectors*[14] k_n separated by $\Delta k = \dfrac{2\pi}{L}$. Hence

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{i k_n x} \frac{L \Delta k}{2\pi}$$

We now let $L \to \infty$[15], so the spacing of wavevectors (and hence frequencies) approaches zero $\Delta k \to 0$, and the discrete set of k_n becomes a continuous variable k, and the infinite sum becomes an integral over k:

$$f(x) \overset{L\to\infty, \Delta k \to 0}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) \mathrm{e}^{ikx}\, \mathrm{d}k$$

where we have defined $\tilde{f}(k) \equiv \lim\limits_{\Delta k \to 0} \dfrac{c_n L}{\sqrt{2\pi}}$. This is exactly the expression in Eqn. 1.11. Therefore, the Fourier transform may be thought of as the generalisation of Fourier series to the case of non-periodic functions.

[13] Leonhard Paul Euler (1707-1783), Swiss mathematician and physicist who made many pioneering contributions.

[14] The wavevector k is related to the wavelength L and frequency f by $k = \dfrac{2\pi}{L} = \dfrac{2\pi f}{c}$, where c is the wave speed.

[15] In other words, the function $f(x)$ is no longer periodic on any finite interval.

1.9.3 Properties of the Fourier Transform

Usually $f(x) = f^*(x)$ is a real function, and we shall assume this to be the case. $\tilde{f}(k)$ is, however, in general complex,

$$\tilde{f}^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(x) \left(\mathrm{e}^{-ikx}\right)^* \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{ikx} \, \mathrm{d}x \neq \tilde{f}(k)$$

If $f(x)$ has definite symmetry, however, then $\tilde{f}(k)$ may be either purely real or purely imaginary,

- If $f(x) = f(-x)$ is an even function $\Rightarrow$ $\tilde{f}^*(k) = \tilde{f}(k)$ is purely real.
- If $f(x) = -f(-x)$ is an odd function $\Rightarrow$ $\tilde{f}^*(k) = -\tilde{f}(k)$ is purely imaginary.

Further useful properties include[16]

- Differentiation.

  $$\mathfrak{F}\left[f^{(n)}(x)\right] = (ik)^n \tilde{f}(k)$$

  In particular

  - $\mathfrak{F}\left[\dfrac{\mathrm{d}f}{\mathrm{d}x}\right] = ik\tilde{f}(k)$
  - $\mathfrak{F}\left[\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right] = -k^2\tilde{f}(k)$

- Multiplication by x.

  $$\mathfrak{F}[xf(x)] = i\frac{\mathrm{d}\tilde{f}(k)}{\mathrm{d}k}$$

- Rigid shift of co-ordinate.

  $$\mathfrak{F}[f(x-a)] = \mathrm{e}^{-ika}\tilde{f}(k)$$

1.9.4 Examples of Fourier Transforms

1. Find the Fourier transform of a square wave

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ -1 & x \in [-1,0] \\ 0 & |x| > 1 \end{cases}$$

[16] Proofs are left as an exercise in the examples sheet.

Exercise: Sketch $f(x)$.

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left[- \int_{-1}^{0} \mathrm{e}^{-ikx}\, \mathrm{d}x + \int_{0}^{1} \mathrm{e}^{-ikx}\, \mathrm{d}x \right] \\
&= \frac{1}{\sqrt{2\pi}} \left\{ - \left[\frac{-1}{ik} \mathrm{e}^{-ikx} \right]_{x=-1}^{0} + \left[\frac{-1}{ik} \mathrm{e}^{-ikx} \right]_{x=0}^{1} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{ik} \left[2 - \left(\mathrm{e}^{ik} + \mathrm{e}^{-ik} \right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{2}{ik} \left(1 - \cos k \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{4}{ik} \sin^2\left(\tfrac{k}{2} \right)
\end{aligned}$$

Note that since $f(x)$ is odd, $\tilde{f}(k)$ is imaginary.

Exercise: Sketch $|\tilde{f}(k)|$.

2. Find the Fourier transform of a *top-hat* function

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}$$

Exercise: Sketch $f(x)$.

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{e}^{-ikx} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{ik} \mathrm{e}^{-ikx} \right]_{x=-a}^{a} \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{ik} \left(\mathrm{e}^{-ika} - \mathrm{e}^{ika} \right) \right] \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin(ka)}{k}
\end{aligned}$$

Note that since $f(x)$ is even, $\tilde{f}(k)$ is real.
Exercise: Sketch $\tilde{f}(k)$.

1.9.5 Bandwidth Theorem

The last example is a good demonstration of the *bandwidth theorem* or *uncertainty principle*: if a function $f(x)$ has a spread Δx in x-space of, then it's spread Δk in k-space will be such that

$$\Delta x \sim \frac{1}{\Delta k}$$

The mathematics here is analogous to that behind the Heisenberg Uncertainty Principle in Quantum Mechanics, which states that for a quantum mechanical particle, such as an electron, its position x and momentum p cannot be both known simultaneously to arbitrary accuracy.

$$\Delta x \geq \frac{h}{\Delta p}$$

where h is Planck's constant.

Another implication is for, eg, laser pulses: the shorter the pulse in time t, the larger the spread $\Delta\omega$ of frequencies of harmonics that constitute it (see example on question sheet).

1.10 Diffraction Through an Aperture

An application of Fourier transforms is in determining diffraction patterns resulting from light incident on an aperture. (The analysis is the same for diffraction of X-rays through crystals.)

It can be shown (though we do not have time to do so here) that the intensity $I(k)$ of light observed in the diffraction pattern is just the square of the Fourier transform of the aperture function $h(x)$:

$$I(k) = |\mathfrak{F}[h(x)]|^2$$

where, refering to Fig. 1.6, $k = \frac{2\pi \sin\theta}{\lambda}$, where λ is the wavelength of the incident light and θ is the angle to the observation point on the screen. For small values of θ we have $\sin\theta \approx \tan\theta = X/D$, hence $k \approx \frac{2\pi X}{\lambda D}$, i.e., k is proportional to the displacement of the observation point on the screen.

Consider diffraction through a single slit of width a as an example. The aperture function is given by a top-hat

$$h(x) = \begin{cases} 1 & x \in [-\frac{a}{2}, \frac{a}{2}] \\ 0 & \text{otherwise} \end{cases}$$

ie, $h = 1$ corresponds to transparent parts of the aperture, and $h = 0$ to opaque parts.

As we showed before, the fourier transform of a top-hat function is given by a sinc function:

$$\tilde{h}(k) = \sqrt{\frac{2}{\pi}} \frac{\sin(ka/2)}{k} = \frac{a\,\mathrm{sinc}(ka/2)}{\sqrt{2\pi}}$$

Hence the intensity is given by

$$I(k) = \frac{a^2\,\mathrm{sinc}^2(ka/2)}{2\pi}.$$

The aperture function and intensity are plotted in Figs. 1.7 and 1.8.

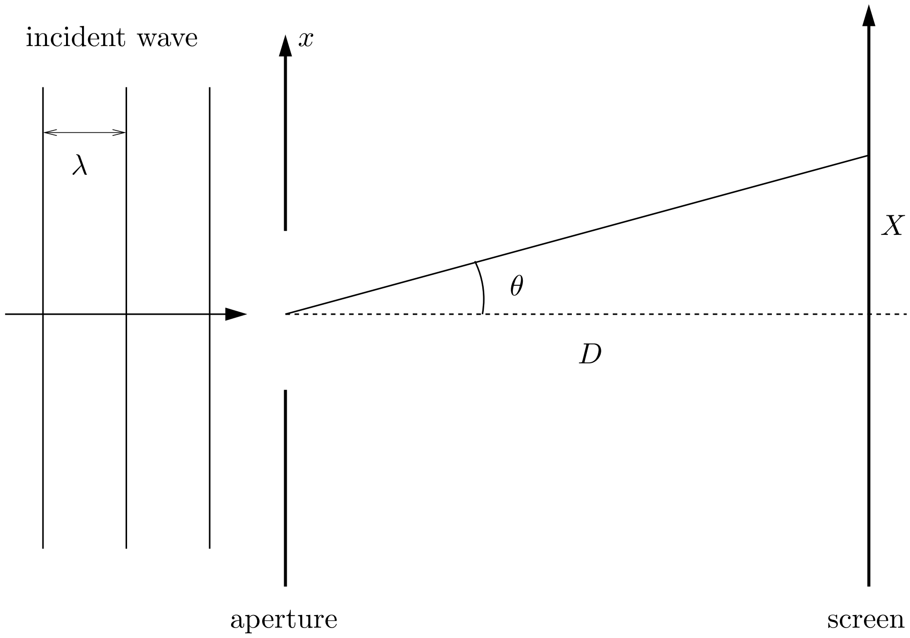


Figure 1.6: Geometry for Fraunhofer diffraction.

1.11 Dirac Delta-Function

As noted earlier, if we Fourier transform a function and then inverse Fourier transform the result, we should get back our original function:

$$f \xrightarrow{\mathfrak{F}} \tilde{f} \xrightarrow{\mathfrak{F}^{-1}} f$$

Let's take Eqn. 1.11 as our starting point and substitute $\tilde{f}(k)$ from Eqn. 1.10 into it

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, \mathrm{e}^{ikx}\, \mathrm{d}k \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\, \mathrm{e}^{-ikx'}\, \mathrm{d}x' \right] \mathrm{e}^{ikx}\, \mathrm{d}k \\
&= \int_{-\infty}^{\infty} f(x') \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ik(x-x')}\, dk \right]}_{\delta(x-x')} dx' \\
\Rightarrow \quad f(x) &= \int_{-\infty}^{\infty} f(x')\, \delta(x-x')\, \mathrm{d}x' \qquad (1.12)
\end{aligned}$$

where we have defined the *Dirac delta-function*[17]

[17]Paul Adrien Maurice Dirac (1902-1984), British theoretical physicist. Dirac shared the Nobel Prize in Physics (1933) with Erwin Schrödinger, for aspects of Quantum Theory.

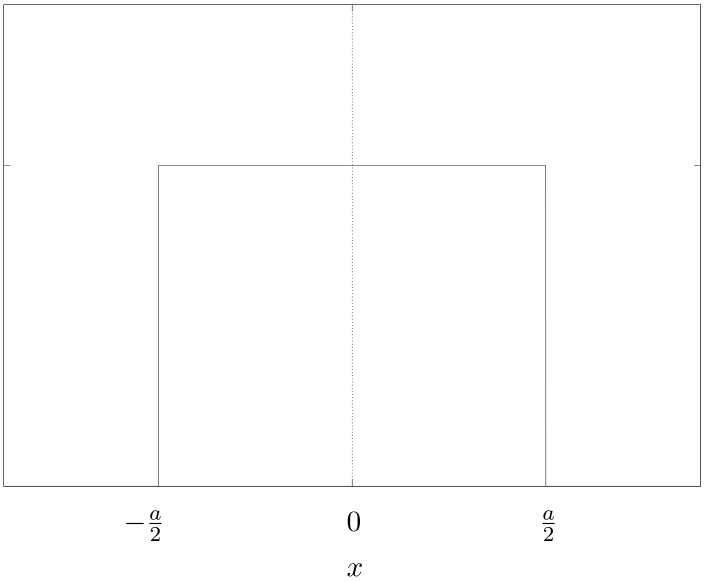


Figure 1.7: Aperture function for single slit of width a.

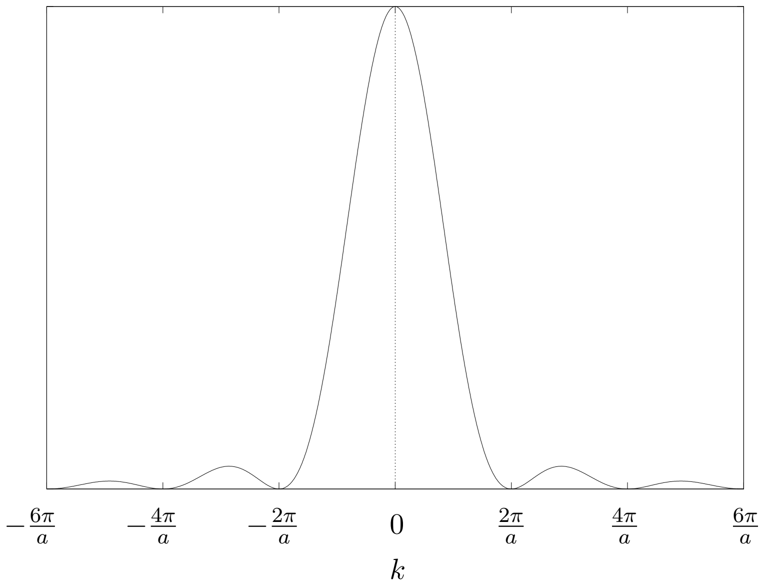


Figure 1.8: Diffraction pattern for single slit of width a: $I(k) = \dfrac{a^2}{2\pi}\mathrm{sinc}^2(ka/2)$.

$$\boxed{\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ik(x-x')}\, dk} \tag{1.13}$$

whose fundamental, defining property is Eqn. 1.12, reproduced here:

$$\boxed{\int_a^b g(x')\,\delta(x-x')\,\mathrm{d}x' = g(x) \qquad \text{if } x \in [a,b]} \tag{1.14}$$

What this shows is that under integration the delta-function "picks out" the value of $g(x)$ at the point on which the delta-function is centred.

1.11.1 Properties of $\delta(x)$

The Dirac delta-function $\delta(x-x')$ is a "spike", centred on $x = x'$, with zero width, infinite height and unit area underneath[18]

$$\delta(x-x') = \begin{cases} \infty & x = x' \\ 0 & \text{otherwise} \end{cases}$$

- $\delta(x) = \delta(-x)$ is an even function
- $\delta^*(x) = \delta(x)$ is a real function
- The area under $\delta(x)$ is 1
$$\int_{-\eta}^{\eta} \delta(x)\,\mathrm{d}x = 1 \quad \forall\, \eta > 0$$
- The Fourier transform of $\delta(x)$ is a constant
$$\tilde{\delta}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)\,\mathrm{e}^{-ikx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-ikx}\Big|_{x=0} = \frac{1}{\sqrt{2\pi}}$$

1.11.2 Understanding $\delta(x)$

$\delta(x)$ as the Limit of a Sinc Function

Keeping Eqn. 1.13 in mind, consider the integral:

$$\delta_\epsilon(x) = \frac{1}{2\pi}\int_{-1/\epsilon}^{1/\epsilon} \mathrm{e}^{ikx}\,\mathrm{d}k$$

[18] Technically, this definition is only heuristic as $\delta(x)$ can not really be called a *function*, rather it is a *distribution* which is defined only under integration, as in Eqn. 1.14.

We can see that

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x)$$

We can integrate the expression for $\delta_\epsilon(x)$:

$$\begin{aligned} \delta_\epsilon(x) &= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{ikx}}{ix}\right]_{k=-1/\epsilon}^{1/\epsilon} \\ &= \frac{1}{\epsilon\pi}\frac{\sin(x/\epsilon)}{x/\epsilon} \\ \Rightarrow \qquad \delta_\epsilon(x) &= \frac{\mathrm{sinc}(x/\epsilon)}{\epsilon\pi} \end{aligned}$$

Where $\mathrm{sinc} x \equiv \dfrac{\sin x}{x}$.

$\delta_\epsilon(x)$ is plotted in Fig. 1.9 for different values of ϵ.

$\delta(x)$ as the Limit of a Top-Hat Function

$\delta(x)$ may be thought of as the limit of a top-hat function of width ϵ and height $\frac{1}{\epsilon}$ as $\epsilon \to 0$:

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x)$$

where

$$\delta_\epsilon(x) = \begin{cases} \epsilon^{-1} & |x| < \epsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

$\delta_\epsilon(x)$ is sketched for different values of ϵ in Fig. 1.10.

Using Fig. 1.11 as a guide, consider the limit

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{x-\epsilon/2}^{x+\epsilon/2} f(x')\,\delta_\epsilon(x-x')\,\mathrm{d}x' &= \lim_{\epsilon\to 0}\int_{x-\epsilon/2}^{x+\epsilon/2} f(x')\,\frac{1}{\epsilon}\,\mathrm{d}x' \\ &= \lim_{\epsilon\to 0} f(a)\int_{x-\epsilon/2}^{x+\epsilon/2} f(x')\,\frac{1}{\epsilon}\,\mathrm{d}x' \quad a \in [x-\frac{\epsilon}{2}, \frac{x+\epsilon}{2}] \\ &= \lim_{\epsilon\to 0} f(a)\left[\frac{x'}{\epsilon}\right]_{x-\epsilon/2}^{x+\epsilon/2} \\ \Rightarrow \int f(x')\,\delta(x-x')\,\mathrm{d}x' &= f(x), \end{aligned}$$

which is exactly Eqn. 1.14.

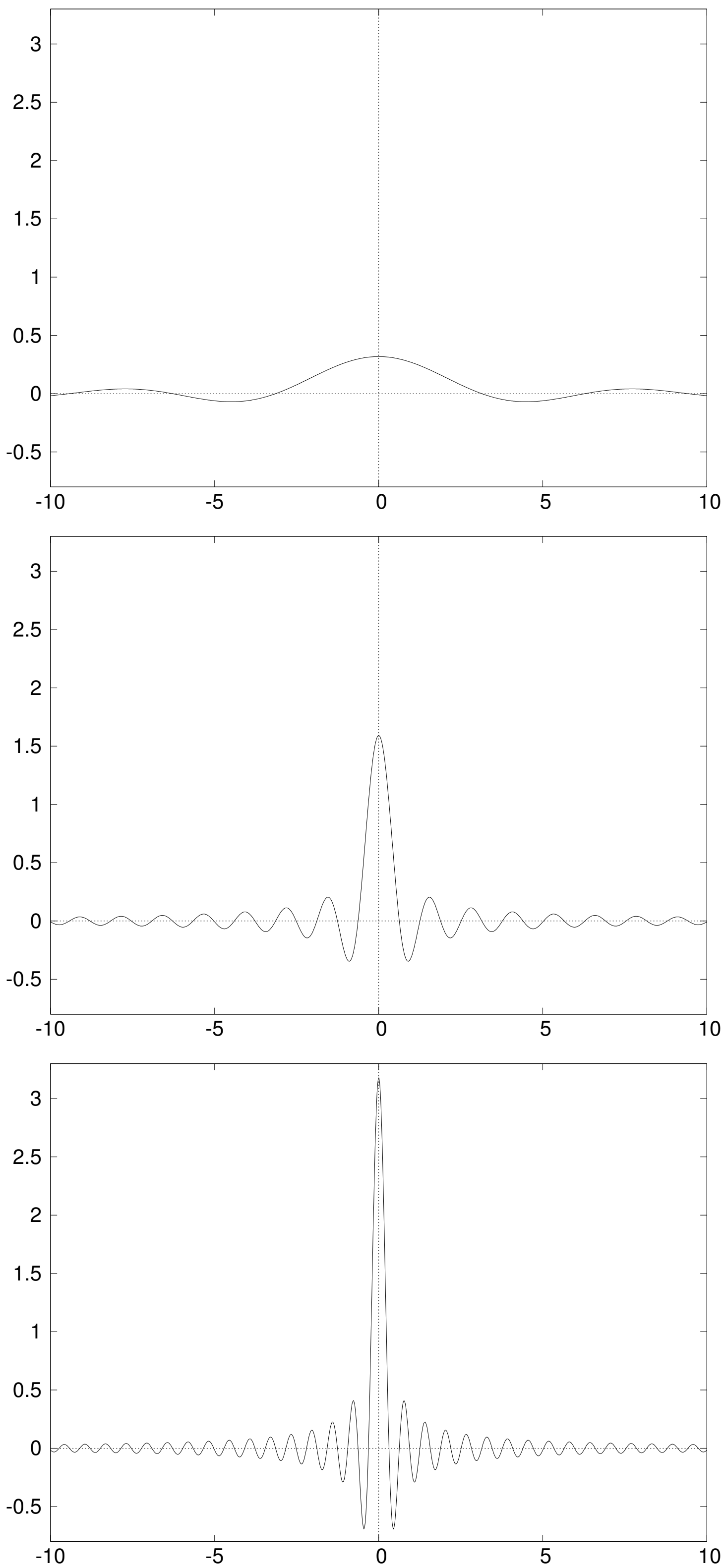


Figure 1.9: $\delta_\epsilon(x) = \dfrac{\mathrm{sinc}(x/\epsilon)}{\epsilon\pi}$ for $\epsilon \in \{1, 0.2, 0.1\}$.

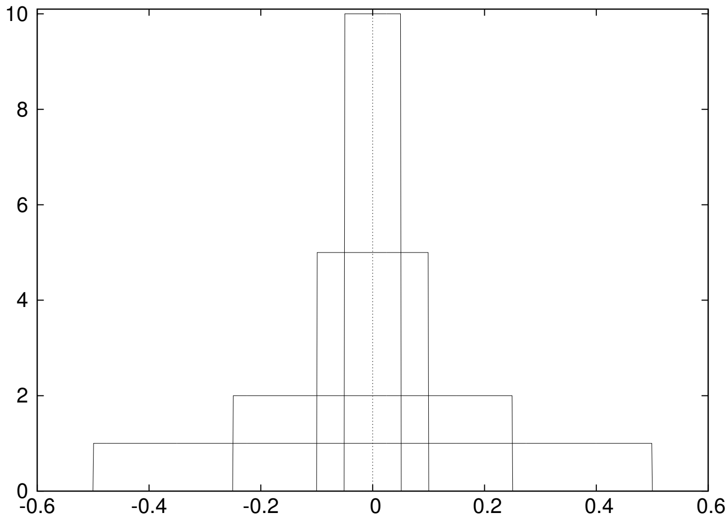


Figure 1.10: $\delta_\epsilon(x)$ for $\epsilon \in \{1, 0.5, 0.2, 0.1\}$.

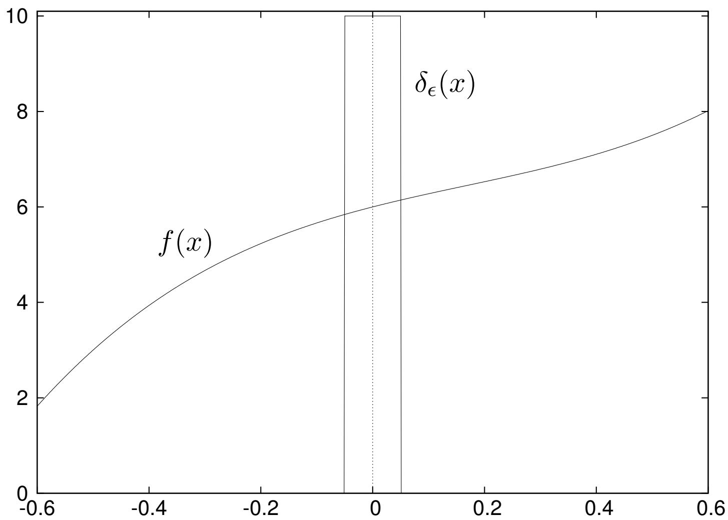


Figure 1.11: Generic sketch of $f(x)$ and $\delta_\epsilon(x)$.

$\delta(x)$ as the Continuous Analogue of δ_{ij}

The Dirac delta-function can also be thought of as the continuous analogue of the discrete Krönecker-delta δ_{ij} which has a property analogous to Eqn. 1.14, but for discrete variables

$$\sum_j \delta_{ij} g_j = g_i.$$

1.12 Parseval's Theorem for Fourier Transforms

Parseval's theorem for Fourier transforms states that

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k} \tag{1.15}$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x &= \int_{-\infty}^{\infty} f(x) f^*(x) \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tilde{f}(k) \mathrm{e}^{ikx} \, \mathrm{d}k \right] \cdot \left[\int_{-\infty}^{\infty} \tilde{f}^*(k') \mathrm{e}^{-ik'x} \, \mathrm{d}k' \right] \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(k') \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i(k-k')x} \, \mathrm{d}x \right]}_{\delta(k-k')} \mathrm{d}k \mathrm{d}k' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(k') \, \delta(k-k') \, \mathrm{d}k \mathrm{d}k' \\
&= \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(k) \, \mathrm{d}k \\
&= \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k.
\end{aligned}$$

The above proof may be easily generalised for two different functions $f(x)$ and $g(x)$:

$$\boxed{\int_{-\infty}^{\infty} f(x) g^*(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{g}^*(k) \, \mathrm{d}k} \tag{1.16}$$

1.13 Convolution Theorem for Fourier Transforms

The *convolution* of two functions $f(x)$ and $g(x)$ is defined as

$$\boxed{f * g \equiv \int_{y=-\infty}^{\infty} f(y) g(x-y) \, \mathrm{d}y} \tag{1.17}$$

from which it is easy to show that $f * g = g * f$: in the above equation, let $\xi = x - y \Rightarrow y = x - \xi,\ \mathrm{d}y = -\mathrm{d}\xi$

$$\begin{aligned} f * g &= \int_{\xi=\infty}^{-\infty} f(x-\xi)g(\xi)\,(-\mathrm{d}\xi) \\ &= \int_{-\infty}^{\infty} g(\xi)f(x-\xi)\,\mathrm{d}\xi \\ \Rightarrow \qquad f * g &= g * f \end{aligned}$$

The *convolution theorem* may be stated in two, equivalent forms. Given two functions f and g,

1. the Fourier transform of their convolution is directly proportional to the product of their Fourier transforms:

$$\boxed{\mathfrak{F}[f * g] = \sqrt{2\pi}\,\tilde{f}\tilde{g}} \tag{1.18}$$

2. the Fourier transform of their product is directly proportional to the convolution of their Fourier transforms:

$$\boxed{\mathfrak{F}[fg] = \frac{1}{\sqrt{2\pi}}\tilde{f} * \tilde{g}} \tag{1.19}$$

The proofs are very similar, so we shall treat one here and the other is left as an exercise in the examples sheet.

Proof:

$$
\begin{aligned}
\mathfrak{F}[f*g] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[f*g](x)\mathrm{e}^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y\right]\mathrm{e}^{-ikx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(y)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-y)\mathrm{e}^{-ikx}\,\mathrm{d}x\right]\mathrm{d}y \qquad \text{(interchanging order of integration)} \\
&= \int_{-\infty}^{\infty} f(y)\mathrm{e}^{-iky}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-y)\mathrm{e}^{-ik(x-y)}\,\mathrm{d}x\right]\mathrm{d}y \qquad (\times \text{ by } \mathrm{e}^{-iky}\mathrm{e}^{iky}=1) \\
&= \int_{-\infty}^{\infty} f(y)\mathrm{e}^{-iky}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi)\mathrm{e}^{-ik\xi}\,\mathrm{d}\xi\right]}_{\tilde{g}(k)}\mathrm{d}y \qquad \text{(substituting } \xi = x-y\text{)} \\
&= \underbrace{\left[\int_{-\infty}^{\infty} f(y)\mathrm{e}^{-iky}\,\mathrm{d}y\right]}_{\sqrt{2\pi}\tilde{f}(k)}\tilde{g}(k) \\
&= \sqrt{2\pi}\,\tilde{f}(k)\tilde{g}(k)
\end{aligned}
$$

Among many other things, the convolution theorem may be used to calculate diffraction patterns of light incident on complicated apertures. In the next section, we will consider the case of Young's double slit experiment.

1.14 Double Slit Diffraction

Earlier it was discussed how, in a Fraunhofer diffraction experiment, the amplitude of the signal at the detection screen is given by the fourier transform of the aperture function $h(x)$. The intensity $I(k)$ that is observed is then just the square of this amplitude:

$$I(k) = \left|\tilde{h}(k)\right|^2$$

where, with reference to Fig. 1.6, $k = \frac{2\pi\sin\theta}{\lambda}$, where λ is the wavelength of the incident wave and θ is the angle of observation. For small values of θ we have $\sin\theta \approx \tan\theta = X/D$, hence $k \approx \frac{2\pi X}{\lambda D}$, i.e., k is proportional to the displacement of the observation point on the screen.

We saw earlier that for a single slit of width a, the aperture function of which is described by a top-hat function, the intensity observed is given by

$$I(k) = \frac{a^2}{2\pi}\mathrm{sinc}^2(ka/2).$$

The aperture function and intensity are plotted in Figs. 1.7 and 1.8.

The aperture function for a double slit is shown in Fig. 1.12. Here the slit separation is d and the slit width is a.

We could go ahead and evaluate the Fourier transform of the aperture function $h(x)$ for the double slit, but there is a more clever approach which uses the convolution theorem.

We note that $h(x)$ is actually the convolution of a double Dirac-delta function (with separation d) with a top-hat function of width a, as shown in Fig. 1.13

$$h(x) = f * g(x).$$

The intensity at the screen is given by

$$I(k) = \left|\tilde{h}(k)\right|^2 = |\mathfrak{F}[f * g]|^2$$

and, from the convolution theorem we know that

$$\mathfrak{F}[f * g] = \sqrt{2\pi}\mathfrak{F}[f]\mathfrak{F}[g]$$

Hence, we can obtain the intensity from the Fourier transforms of $f(x)$ and $g(x)$. $g(x)$ is a top-hat, and as we've seen before, it is easy to show that

$$\mathfrak{F}[g] = \frac{a}{\sqrt{2\pi}}\mathrm{sinc}(ka/2)$$

For the double Dirac-delta function, we see that

$$f(x) = \delta(x - d/2) + \delta(x + d/2)$$

hence

$$\begin{aligned}
\mathfrak{F}[f] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\delta(x - d/2) + \delta(x + d/2)\right]\mathrm{e}^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\left[\mathrm{e}^{-ikd/2} + \mathrm{e}^{ikd/2}\right] \\
&= \sqrt{\frac{2}{\pi}}\cos(kd/2)
\end{aligned}$$

Putting it all together we see that the intensity on the screen is given by

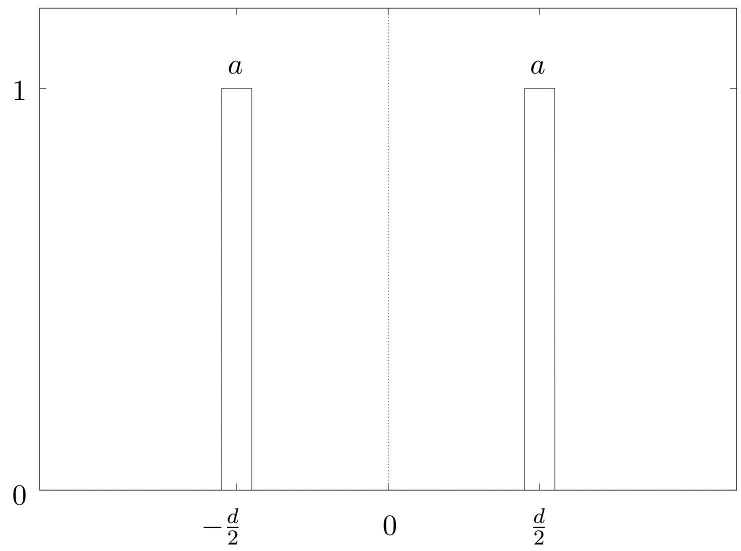


Figure 1.12: Aperture function for double slit: slit width a, separation d.

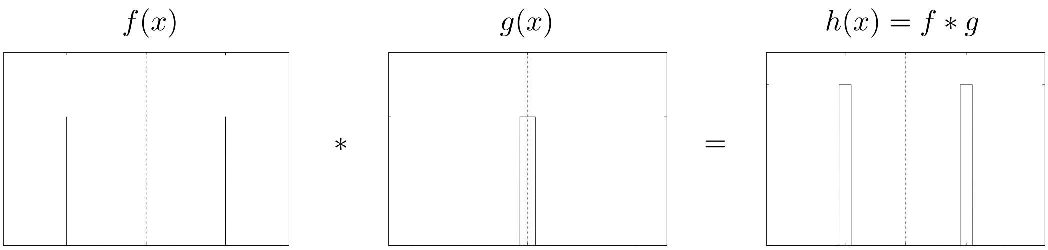


Figure 1.13: Convolution theorem for double slit.

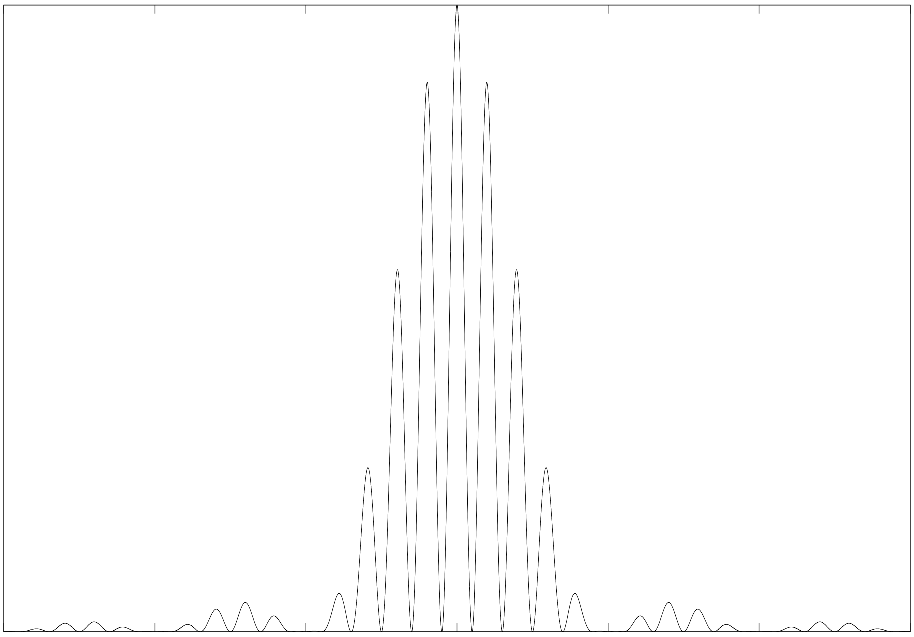


Figure 1.14: Diffraction pattern for double slit.

$$I(k) = \frac{2a^2}{\pi} \mathrm{sinc}^2(ka/2)\cos^2(ka/2)$$

which is plotted in Fig. 1.14.

What is seen is the broad envelope function which comes from the sinc$(ka/2)$ factor, and the denser interference fringes that come from the $\cos(kd/2)$ factor. Note that $d > a$, which is why the frequency of the fringes is higher than that of the envelope function (a manifestation of the bandwidth theorem that was discussed in a previous lecture).

1.15 Differential Equations

One important use of Fourier transforms is in solving differential equations. A Fourier transform may be used to transform the a differential equation into an algebraic equation since

$$\mathfrak{F}\left[\frac{\partial^n f}{\partial x^n}\right] = (ik)^n \mathfrak{F}[f].$$

1.15.1 Example: Heat Diffusion Equation

Consider an infinite, one-dimensional bar which has an initial temperature distribution given by $\theta(x, t = 0) = \delta(x)$. This is clearly a somewhat unphysical situation, but it's not such a bad approximation to having a situation in which a lot of heat is initially concentrated at the mid-point of the rod.

The flow of heat is determined by the diffusion equation

$$D\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}$$

which is second-order in x and first-order in t. D is the diffusion coefficient. The boundary conditions on this problem are that $\theta(\pm\infty, t) = 0$.

Consider what happens if we Fourier transform both sides of this equation with respect to the variable x:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} D\frac{\partial^2\theta}{\partial x^2}\mathrm{e}^{-ikx}\,\mathrm{d}x &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial\theta}{\partial t}\mathrm{e}^{-ikx}\,\mathrm{d}x \\
D\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2\theta}{\partial x^2}\mathrm{e}^{-ikx}\,\mathrm{d}x &= \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\theta(x,t)\mathrm{e}^{-ikx}\,\mathrm{d}x\right] \\
D\mathfrak{F}\left[\frac{\partial^2\theta(x,t)}{\partial x^2}\right] &= \frac{\partial}{\partial t}\mathfrak{F}[\theta(x,t)] \\
-Dk^2\tilde{\theta}(k,t) &= \frac{\partial\tilde{\theta}(k,t)}{\partial t}
\end{aligned}$$

So the differential equation in x is replaced by an algebraic equation in k. We still have a differential equation in t which is first-order, and also easy to solve:

$$\tilde{\theta}(k,t) = A(k)\mathrm{e}^{-Dk^2t}$$

where $A(k) = \tilde{\theta}(k, t = 0)$, a constant of integration, is just the Fourier transform of the initial condition $\theta(x, t = 0) = \delta(x)$, which we can calculate:

$$A(k) = \tilde{\theta}(k, t = 0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)\mathrm{e}^{-ikx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}.$$

Hence,

$$\tilde{\theta}(k,t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-Dk^2t}$$

So, we have Fourier transformed the original differential equation from "x-space" to "k-space", solved it in k-space, and applied the initial condition in k-space. All that remains to be done is to inverse Fourier transform $\tilde{\theta}(k,t)$ back in order to obtain the solution in x-space.

In practice, this is typically the most difficult part of the process (there's no such thing as a free lunch!), but for the example we have chosen, it turns out to be not too bad because, through completing the square and changing the variable of integration, we are able to reduce the problem to a Gaussian integral of known value[19]:

$$\begin{aligned}
\theta(x,t) &= \mathfrak{F}^{-1}[\tilde{\theta}(k,t)] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-Dk^2t}\mathrm{e}^{ikx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-Dt\left[(k-ix/2Dt)^2\right]-x^2/4Dt}\,\mathrm{d}x \qquad \text{combining exponents and completing the square} \\
&= \frac{1}{2\pi}\mathrm{e}^{-x^2/4Dt}\int_{-\infty}^{\infty} \mathrm{e}^{-Dtq^2}\,\mathrm{d}q \qquad \text{changing variable to } q = k - ix/2Dt \Rightarrow \mathrm{d}q = \mathrm{d}k \\
&= \frac{1}{2\pi}\sqrt{\frac{\pi}{Dt}}\mathrm{e}^{-x^2/4Dt} \qquad \text{using the standard result for a Gaussian integral}
\end{aligned}$$

Hence the final result[20]

$$\theta(x,t) = \frac{1}{2\sqrt{\pi Dt}}\mathrm{e}^{-x^2/4Dt}$$

Exercise: Verify that $\theta(x,t)$ satisfies the original differential equation and boundary conditions.

Fig. 1.15 shows the temperature distribution for different times t. Is the behaviour as you would expect?

[19] $\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2}\,\mathrm{d}x = \sqrt{\pi/\alpha}$.

[20] The observant will have noticed that, in the fourth line of the derivation above, I should also have changed the limits on the integral to $q = \pm\infty - ix/2Dt$, thus taking the contour of the integral off the real-q axis. It turns out that I'm allowed to shift the contour back to the real-axis since the integrand has no singularities anywhere. This is the subject of fields of mathematics known as *complex analysis* and *contour integration*, which have many uses in physics and engineering, but which we won't cover in the course this year.

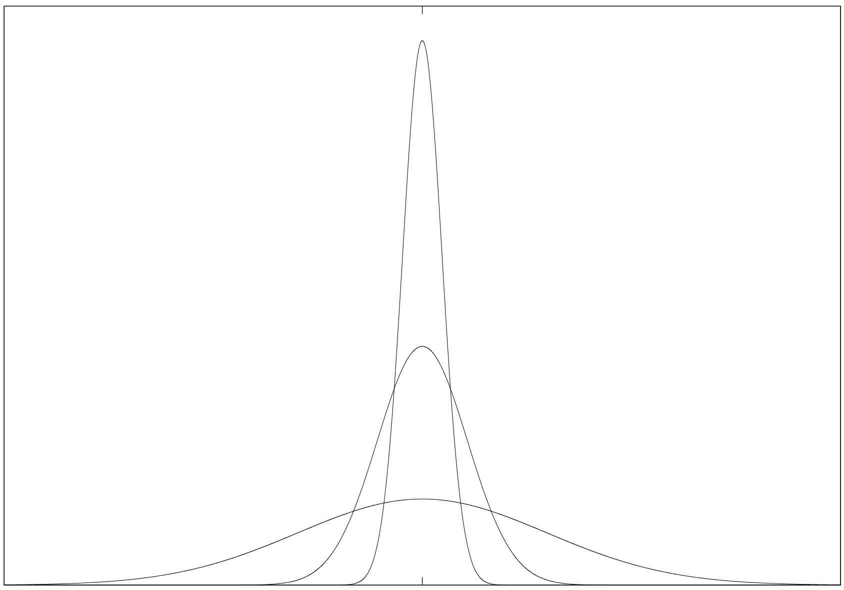


Figure 1.15: $\theta(x, t)$ for three different values of t.