QUESTION 2

$$q^{n-1}f\left(\frac{p}{q}\right)=0 \Rightarrow \frac{p^n}{q}+a_{n-1}p^{n-1}+a_{n-2}p^{n-2}q+a_{n-3}p^{n-3}q^2+\ldots+a_1pq^{n-2}+a_0q^{n-1}=0$$

Every term except the first is an integer.

For the expression to equal zero $\frac{p^n}{q}$ must be an integer.

Since p and q have no common factors greater than 1 $q=1$.

It is always possible, for <u>any</u> rational number, to find an equivalent rational number in the form defined for $\frac{p}{q}$.

Therefore if <u>any</u> rational number is a root of a polynomial equation of the given form, we can replace it in the equation by this form and apply the same reasoning as already given to show that the $\frac{p}{q}$ form must be an integer. If this is true then the original rational number must itself have been an integer (albeit in disguised form).

Part (i)

The n^{th} root of of 2 satisfies the equation $x^n-2=0$ for $n\geq 2$.

This equation has integer coefficients and so is of the form of $f(x)$ (given).

By the given reasoning this means that if the solution were a rational number then it would be an integer.

This is clearly impossible.

Therefore the n^{th} root of of 2 is irrational ($n\geq 2$).

Part (ii)

Suppose $x^3 - x + 1 = 0$ has a rational solution.

This equation has integer coefficients and so is of the form of $f(x)$ (given).

By the given reasoning this means that if the solution were a rational number then it would be an integer.

Now $x^3 - x + 1 = 0 \Rightarrow 1 = x(1 - x^2)$

Clearly $x \notin \{-1, 0, 1\}$

Further for all x such that $x \in \mathbb{Z}$ and $|x| > 1$ we have $x^2 > 1$ so that $|x(1 - x^2)| > 1$

Therefore there are no integer solutions to the equation.

Therefore, by our original theorem there are no rational solutions.

Part (iii)

Suppose $x^n - 5x + 7 = 0$ for $n \geq 2$ has a rational solution.

This equation has integer coefficients and so is of the form of $f(x)$ (given).

By the given reasoning this means that if the solution were a rational number then it would be an integer.

Now $x^n - 5x + 7 = 0 \Rightarrow 7 = x(5 - x^{n-1})$

Since x is an integer and 7 is prime $x = 1$ or 7

Clearly $x = 1 \Rightarrow x(1 - x^{n-1}) = 0 \neq 7$ and $x = 7 \Rightarrow 7(1 - 7^{n-1}) \neq 5$ for any n

Therefore there is no integer solution and thus no rational solution to the equation.