1a)

$$f(x)=\frac{3x+1}{x^2+1}$$
$$u=3x+1,\ v=x^2+1$$

$$f'(x)=\frac{u'v-uv'}{v^2}$$
$$=\frac{3(x^2+1)-2x(3x+1)}{(x^2+1)^2}$$
$$=\frac{3x^2+3-6x^2-2x}{(x^2+1)^2}$$
$$=\frac{-3x^2-2x+3}{(x^2+1)^2}$$

b)

$$g(x)=\cos^2 x\exp(\tan x)$$

$$\ln g(x)=2\ln(\cos x)+\tan x$$
$$\frac{g'(x)}{g(x)}=\frac{-2\sin x}{\cos x}+\sec^2 x$$
$$g'(x)=g(x)\left(\frac{-2\sin x}{\cos x}+\sec^2 x\right)$$
$$=\cos^2 x\exp(\tan x)\left(\frac{-2\sin x}{\cos x}+\sec^2 x\right)$$
$$=\exp(\tan x)(1-2\sin x\cos x)$$
$$=\exp(\tan x)(1-\sin 2x)$$

2)

$$g_1=a=2048$$
$$g_4=ar^3=256$$
$$r^3=\frac{256}{2048}=\frac{1}{8}$$
$$r=\frac{1}{2}$$

2contd.)

$$S_n=\frac{a(1-r^n)}{1-r}$$
$$=\frac{2048\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}=4088$$

$$1-\frac{1}{2^n}=\frac{2044}{2048}$$
$$\frac{1}{2^n}=\frac{2048-2044}{2048}$$
$$=\frac{4}{2048}=\frac{1}{512}=\frac{1}{2^9}$$

$$n=9$$

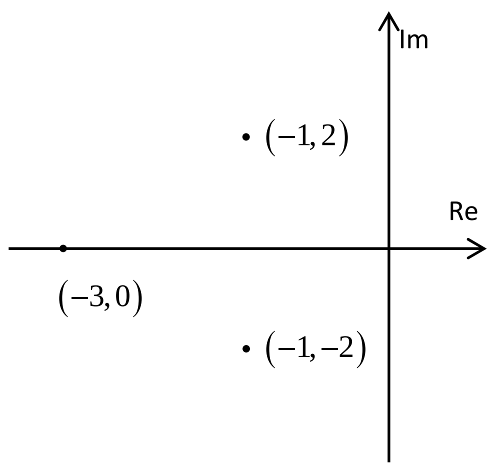
3)

$(2i-1)$ is a root $\Rightarrow$ $(z-2i+1)$ is a factor
$\Rightarrow$ $(z+2i+1)$ is also a factor

$$(z-2i+1)(z+2i+1)=z^2+2z+5$$

$$\begin{array}{r|rrrr}
 & & & z & +3 \\
z^2+2z+5 & z^3 & +5z^2 & +11z & +15 \\
 & z^3 & +2z^2 & +5z & \\
 & & 3z^2 & +6z & +15 \\
 & & 3z^2 & +6z & +15 \\
 & & & & 0
\end{array}$$

$$z=2i-1,\ -2i-1,\ -3$$



4)

$$\binom{9}{r}(2x)^{9-r}\left(-\frac{1}{x^2}\right)^r = \binom{9}{r}\left(2^{9-r}\right)\left(x^{9-r}\right)(-1)^r\left(x^{-2r}\right)$$
$$= \binom{9}{r}(-1)^r\left(2^{9-r}\right)\left(x^{9-3r}\right)$$

$$9-3r=0$$
$$r=3$$
$$\binom{9}{r}(-1)^r\left(2^{9-r}\right) = -2^6\binom{9}{3}$$
$$=-5376$$

5)

$$\overrightarrow{PQ} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-2\\1\\-1\end{pmatrix} = \begin{pmatrix}3\\1\\4\end{pmatrix}$$
$$\overrightarrow{PR} = \begin{pmatrix}3\\0\\1\end{pmatrix} - \begin{pmatrix}-2\\1\\-1\end{pmatrix} = \begin{pmatrix}5\\-1\\2\end{pmatrix}$$

$$\mathbf{n} = \overrightarrow{PQ}\times\overrightarrow{PR}$$
$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 4\\ 5 & -1 & 2\end{vmatrix}$$
$$= \mathbf{i}(2+4)-\mathbf{j}(6-20)+\mathbf{k}(-3-5)$$
$$= 6\mathbf{i}+14\mathbf{j}-8\mathbf{k}$$

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$$
$$6x+14y-8z = (6\mathbf{i}+14\mathbf{j}-8\mathbf{k})\cdot\begin{pmatrix}-2\\1\\-1\end{pmatrix}$$
$$= -12+14+8 = 10$$

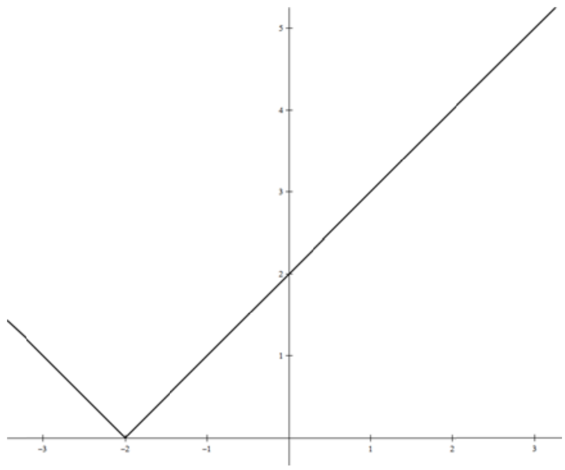
$$6x+14y-8z=10$$
$$3x+7y-4z=5$$

6)

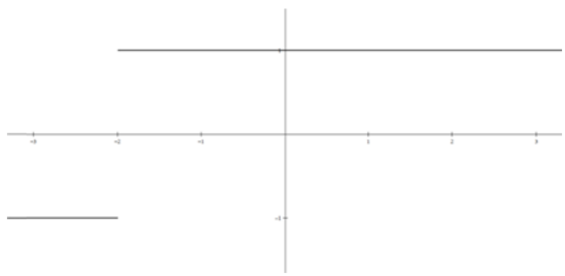
$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots$$

$$\left(1+e^x\right)^2 = \left(1+1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots\right)^2$$
$$= \left(2+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots\right)^2$$
$$= 4+2x+x^2+\frac{x^3}{3}+2x+x^2+\frac{x^3}{2}+x^2+\frac{x^3}{2}+\frac{x^3}{3}+\ldots$$
$$= 4+4x+3x^2+\frac{5x^3}{3}+\ldots$$

7a)



b)



8)

$$I=\int_0^2\sqrt{16-x^2}\,dx$$

$$x=4\sin\theta$$
$$dx=4\cos\theta d\theta$$

$$x=2,\ \ \theta=\frac{\pi}{6}$$
$$x=0,\ \ \theta=0$$

$$\begin{aligned}I&=\int_0^{\pi/6}4\cos\theta\sqrt{16-16\sin^2\theta}\,d\theta\\&=16\int_0^{\pi/6}\cos\theta\sqrt{1-\sin^2\theta}\,d\theta\\&=16\int_0^{\pi/6}\cos^2\theta\,d\theta\\&=16\int_0^{\pi/6}\left(\frac{1}{2}\cos2\theta+\frac{1}{2}\right)d\theta\\&=8\left[\frac{\sin2\theta}{2}+\theta\right]_0^{\pi/6}\\&=8\left(\frac{1}{2}\sin\frac{\pi}{3}+\frac{\pi}{6}-0\right)\\&=2\sqrt{3}+\frac{4\pi}{3}\end{aligned}$$

9)

$$\begin{aligned}\mathbf{A}+\mathbf{A}^{-1}&=\mathbf{I}\\\mathbf{AA}+\mathbf{AA}^{-1}&=\mathbf{AI}\\\mathbf{A}^2+\mathbf{I}&=\mathbf{A}\\\mathbf{A}^2+\mathbf{A}+\mathbf{A}^{-1}&=\mathbf{A}\\\mathbf{A}^2+\mathbf{A}^{-1}&=\mathbf{0}\\\mathbf{AA}^2+\mathbf{AA}^{-1}&=\mathbf{A0}\\\mathbf{A}^3+\mathbf{I}&=\mathbf{0}\\\mathbf{A}^3&=-\mathbf{I}\end{aligned}$$

$$k=-1$$

10)

$$\begin{array}{r}3\ \text{r}4\\7\overline{\big)\ 25\ \text{r}1}\\7\overline{\big)\ 176\ \text{r}2}\\7\overline{\big)\ 1234}\end{array}$$

Hence $1234_{10}=3412_7$

11a)

$$\frac{d\left(\sin^{-1}x\right)}{dx}=\frac{1}{\sqrt{1-x^2}}$$

b)

$$\begin{aligned}\int\frac{xdx}{\sqrt{1-x^2}}&=\int\frac{-1}{2}\times(-2x)\left(1-x^2\right)^{-1/2}dx\\&=-\sqrt{1-x^2}+C\end{aligned}$$

$$I=\int\frac{x\sin^{-1}xdx}{\sqrt{1-x^2}}$$

$$u=\sin^{-1}x\qquad v'=\frac{x}{\sqrt{1-x^2}}$$
$$u'=\frac{1}{\sqrt{1-x^2}}\qquad v=-\sqrt{1-x^2}$$

$$\begin{aligned}I&=uv-\int u'v\\&=-\sin^{-1}x\sqrt{1-x^2}+\int\frac{\sqrt{1-x^2}dx}{\sqrt{1-x^2}}\\&=-\sin^{-1}x\sqrt{1-x^2}+\int dx\\&=x-\sin^{-1}x\sqrt{1-x^2}+C\end{aligned}$$

12)

$$\frac{dh}{dt} = 0.01$$
$$\frac{dr}{dt} = -0.02$$

$$V = \pi r^2 h$$
$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$$
$$= \pi r \left(r\frac{dh}{dt} + 2h\frac{dr}{dt} \right)$$
$$= 0.6\pi \left(0.6 \times 0.01 - 2\times 2\times 0.02\right)$$
$$= 0.6\pi \times \left(-0.074\right)$$
$$= -0.0444\pi \text{ m}^3\text{s}^{-1}$$

13)

$$\frac{dx}{dt} = 2 + t$$
$$\frac{dy}{dt} = t^2 - 3$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2 - 3}{2 + t}$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{2t\left(2+t\right) - \left(t^2 - 3\right)}{\left(2+t\right)^2}$$
$$= \frac{4t + 2t^2 + 3 - t^2}{\left(2+t\right)^2}$$
$$= \frac{t^2 + 4t + 3}{\left(2+t\right)^2}$$
$$= \frac{\left(t+3\right)\left(t+1\right)}{\left(2+t\right)^2}$$
$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \times \frac{dt}{dx} = \frac{\left(t+3\right)\left(t+1\right)}{\left(2+t\right)^3}$$

13 contd.)

For SPs, $\frac{dy}{dx} = \frac{t^2-3}{2+t} = 0$

$$t^2 - 3 = 0$$
$$t = \pm\sqrt{3}$$

$t = \sqrt{3}$:

$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{3}+3\right)\left(\sqrt{3}+1\right)}{\left(2+\sqrt{3}\right)^3} > 0, \text{ Minimum at } t = \sqrt{3}$$

$t = -\sqrt{3}$:

$$\frac{d^2y}{dx^2} = \frac{\left(3-\sqrt{3}\right)\left(1-\sqrt{3}\right)}{\left(2-\sqrt{3}\right)^3}$$

$1 < \sqrt{3} < 2$ so only $1-\sqrt{3}$ is negative

hence $\frac{d^2y}{dx^2} < 0$, Maximum at $t = -\sqrt{3}$

For inflexion points $\frac{d^2y}{dx^2} = 0$:

$$\frac{\left(t+3\right)\left(t+1\right)}{\left(2+t\right)^3} = 0$$
$$\left(t+3\right)\left(t+1\right) = 0$$
$$t = -1, -3$$

$$t = -1: \left(\frac{1}{2} - 2, 3 - \frac{1}{3}\right) = \left(\frac{-3}{2}, \frac{8}{3}\right)$$
$$t = -3: \left(-6 + \frac{9}{2}, -9 + 9\right) = \left(\frac{-3}{2}, 0\right)$$

Hence two distinct points

14a)

$$\left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 2 & -2 & 4 & -1 \\ -1 & 1 & \lambda & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 4 & -1 \\ 4 & 0 & 6 & 1 \\ -1 & 1 & \lambda & 2 \end{array}\right) \quad (\text{swap } R_2 \text{ and } R_1)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 4 & -1 \\ 0 & 4 & -2 & 3 \\ -1 & 1 & \lambda & 2 \end{array}\right) \quad (R_2 - 2R_1)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 4 & -1 \\ 0 & 4 & -2 & 3 \\ 0 & 0 & 2\lambda+4 & 3 \end{array}\right) \quad (2R_3 + R_1)$$

$$(2\lambda+4)z = 3$$
$$z = \frac{3}{2(\lambda+2)}$$

$$4y - 2z = 3$$
$$4y = 3 + \frac{3}{\lambda+2} = \frac{3\lambda+9}{\lambda+2}$$
$$y = \frac{3\lambda+9}{4(\lambda+2)}$$

$$2x - 2y + 4z = -1$$
$$2x = -1 + 2y - 4z$$
$$= -1 + \frac{3\lambda+9}{2(\lambda+2)} - \frac{12}{2\lambda+4}$$
$$= \frac{3\lambda - 3 - 2\lambda - 4}{2(\lambda+2)}$$
$$= \frac{\lambda-7}{2(\lambda+2)}$$
$$x = \frac{\lambda-7}{4(\lambda+2)}$$

b)

$$\lambda = -2:$$
$$(2\lambda+4)z = 3$$
$$0z = 0 = 3$$

System is inconsistent so no solutions

c)

$$\lambda = -2.1:$$
$$x = \frac{-2.1-7}{4(2-2.1)} = 22.75$$
$$y = \frac{-6.3+9}{4(2-2.1)} = -6.75$$
$$z = \frac{3}{2(2-2.1)} = -15$$

A small change in λ resulted in a large change in the solutions. The system is ill-conditioned.

15a)

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

$$x = -2:$$
$$1 = 0 + 0 + C(-2-1) = -3C$$
$$C = -\frac{1}{3}$$

$$x = 1:$$
$$1 = A(1+2)^2 + 0 + 0 = 9A$$
$$A = \frac{1}{9}$$

Comparing coefficients on x^2 :
$$0 = A + B$$
$$B = -\frac{1}{9}$$

$$\frac{1}{(x-1)(x+2)^2} = \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$$

b)

$$(x-1)\frac{dy}{dx} - y = \frac{x-1}{(x+2)^2}$$
$$\frac{dy}{dx} - \frac{y}{x-1} = \frac{1}{(x+2)^2}$$

$$I(x) = \exp\left(-\int \frac{dx}{x-1}\right) = \frac{1}{x-1}$$
$$\frac{1}{x-1}\left(\frac{dy}{dx} - \frac{y}{x-1}\right) = \frac{1}{(x-1)(x+2)^2} = \frac{d\left(yI(x)\right)}{dx}$$
$$yI(x) = \int \frac{dx}{(x-1)(x+2)^2}$$
$$= \int \left[\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}\right] dx$$
$$= \frac{1}{9}\ln(x-1) - \frac{1}{9}\ln(x+2) + \frac{1}{3(x+2)} + C$$
$$= \frac{1}{9}\ln\left(\frac{x-1}{x+2}\right) + \frac{1}{3(x+2)} + C$$
$$y = (x-1)\left[\frac{1}{9}\ln\left(\frac{x-1}{x+2}\right) + \frac{1}{3(x+2)} + C\right]$$

16a)

Base case $n=1$:

$$\begin{aligned}(\cos\theta+i\sin\theta)^1 &= \cos\theta+i\sin\theta\\ &= \cos 1\theta+i\sin 1\theta\end{aligned}$$

Hence true for the base case.

Assume true for some integer $n=k\geq 1$, so:

$$(\cos\theta+i\sin\theta)^k=\cos k\theta+i\sin k\theta$$

Then for $n=k+1$:

$$\begin{aligned}(\cos\theta+i\sin\theta)^n &= (\cos\theta+i\sin\theta)^{k+1}\\ &= (\cos\theta+i\sin\theta)^k(\cos\theta+i\sin\theta)\\ &= (\cos k\theta+i\sin k\theta)(\cos\theta+i\sin\theta)\\ &= \cos k\theta\cos\theta-\sin k\theta\sin\theta\\ &\quad +i\sin k\theta\cos\theta+i\cos k\theta\sin\theta\\ &= (\cos k\theta\cos\theta-\sin k\theta\sin\theta)\\ &\quad +i(\sin k\theta\cos\theta+\cos k\theta\sin\theta)\\ &= \cos(k\theta+\theta)+i\sin(k\theta+\theta)\\ &= \cos(k+1)\theta+i\sin(k+1)\theta\\ &= \cos n\theta+i\sin n\theta\end{aligned}$$

Hence true for $n=k+1$ when true for $n=k$.

Since the statement is also true for the base case $n=1$, by induction it is true for all integer $n\geq 1$.

b)

$$A=\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^{11}=\cos\frac{11\pi}{18}+i\sin\frac{11\pi}{18}$$

$$B=\left(\cos\frac{\pi}{36}+i\sin\frac{\pi}{36}\right)^{4}=\cos\frac{4\pi}{36}+i\sin\frac{4\pi}{36}$$

$$\arg\left[\frac{A}{B}\right]=\arg A-\arg B=\frac{11\pi}{18}-\frac{4\pi}{36}=\frac{18\pi}{36}=\frac{\pi}{2}$$

Hence $\frac{A}{B}$ is on the imaginary axis and therefore its real part is zero.