OCR
RECOGNISING ACHIEVEMENT

Friday 20 January 2012 – Afternoon

A2 GCE MATHEMATICS (MEI)

4753/01 Methods for Advanced Mathematics (C3)

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4753/01
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

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INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **8** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Section A (36 marks)

1 Differentiate $x^2 \tan 2x$. **[3]**

2 The functions $f(x)$ and $g(x)$ are defined as follows.

$$f(x) = \ln x, \qquad x > 0$$
$$g(x) = 1 + x^2, \qquad x \in \mathbb{R}$$

Write down the functions $fg(x)$ and $gf(x)$, and state whether these functions are odd, even or neither. **[4]**

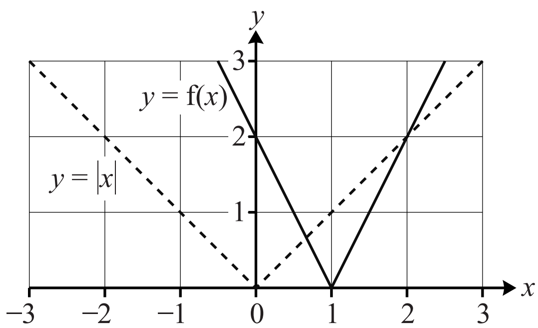
3 Show that $\int_0^{\frac{\pi}{2}} x \cos\frac{1}{2}x \, dx = \frac{\sqrt{2}}{2}\pi + 2\sqrt{2} - 4$. **[5]**

4 Prove or disprove the following statement:

'No cube of an integer has 2 as its units digit.' **[2]**

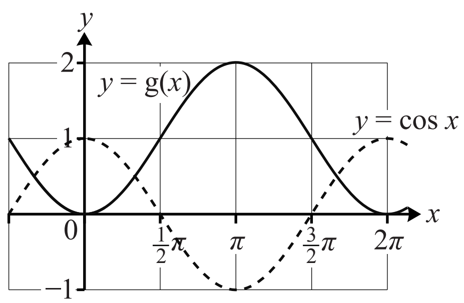
5 Each of the graphs of $y = f(x)$ and $y = g(x)$ below is obtained using a sequence of two transformations applied to the corresponding dashed graph. In each case, state suitable transformations, and hence find expressions for $f(x)$ and $g(x)$.

(i)



[3]

(ii)



[3]

6 Oil is leaking into the sea from a pipeline, creating a circular oil slick. The radius r metres of the oil slick t hours after the start of the leak is modelled by the equation

$$r = 20(1 - e^{-0.2t}).$$

(i) Find the radius of the slick when $t = 2$, and the rate at which the radius is increasing at this time. **[4]**

(ii) Find the rate at which the area of the slick is increasing when $t = 2$. **[4]**

7 Fig. 7 shows the curve $x^3 + y^3 = 3xy$. The point P is a turning point of the curve.

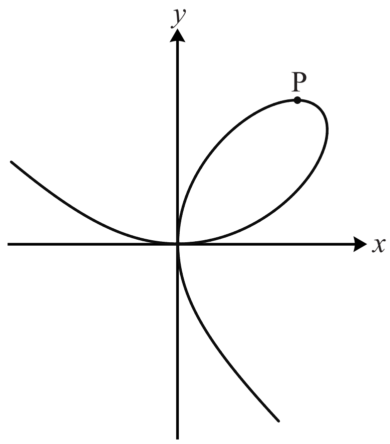


Fig. 7

(i) Show that $\dfrac{dy}{dx} = \dfrac{y - x^2}{y^2 - x}$. **[4]**

(ii) Hence find the exact x-coordinate of P. **[4]**

Section B (36 marks)

8 Fig. 8 shows the curve $y = \dfrac{x}{\sqrt{x-2}}$, together with the lines $y = x$ and $x = 11$. The curve meets these lines at P and Q respectively. R is the point (11, 11).

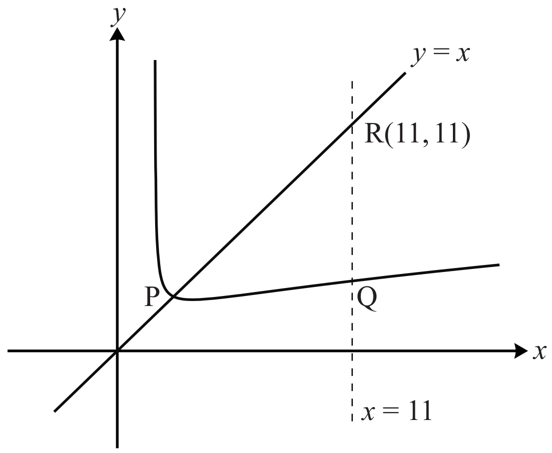


Fig. 8

(i) Verify that the x-coordinate of P is 3. **[2]**

(ii) Show that, for the curve, $\dfrac{dy}{dx} = \dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$.

Hence find the gradient of the curve at P. Use the result to show that the curve is **not** symmetrical about $y = x$. **[7]**

(iii) Using the substitution $u = x - 2$, show that $\displaystyle\int_3^{11} \frac{x}{\sqrt{x-2}}\,dx = 25\tfrac{1}{3}$.

Hence find the area of the region PQR bounded by the curve and the lines $y = x$ and $x = 11$. **[9]**

9 Fig. 9 shows the curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$. The function $y = \mathrm{f}(x)$ is given by

$$\mathrm{f}(x) = \ln\left(\frac{2x}{1+x}\right),\ x > 0.$$

The curve $y = \mathrm{f}(x)$ crosses the x-axis at P, and the line $x = 2$ at Q.

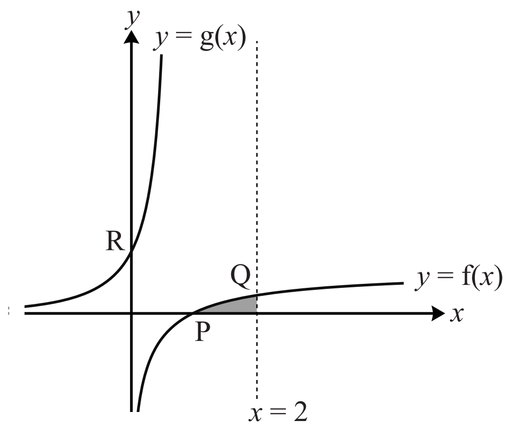


Fig. 9

(i) Verify that the x-coordinate of P is 1.

Find the exact y-coordinate of Q. **[2]**

(ii) Find the gradient of the curve at P. [Hint: use $\ln \frac{a}{b} = \ln a - \ln b$.] **[4]**

The function $\mathrm{g}(x)$ is given by

$$\mathrm{g}(x) = \frac{\mathrm{e}^x}{2 - \mathrm{e}^x}, \quad x < \ln 2.$$

The curve $y = \mathrm{g}(x)$ crosses the y-axis at the point R.

(iii) Show that $\mathrm{g}(x)$ is the inverse function of $\mathrm{f}(x)$.

Write down the gradient of $y = \mathrm{g}(x)$ at R. **[5]**

(iv) Show, using the substitution $u = 2 - \mathrm{e}^x$ or otherwise, that $\int_0^{\ln\frac{4}{3}} \mathrm{g}(x)\,\mathrm{d}x = \ln \frac{3}{2}$.

Using this result, show that the exact area of the shaded region shown in Fig. 9 is $\ln \frac{32}{27}$.
[Hint: consider its reflection in $y = x$.] **[7]**

Question			Answer	Marks	Guidance	
1			$y = x^2 \tan 2x$ $\Rightarrow$ dy/dx = $2x^2\sec^2 2x + 2x\tan 2x$	M1 M1 A1cao	product rule d/du(tan u) = $\sec^2 u$ soi or $2x^2/\cos^2 2x + 2x\tan 2x$	u × their v' + v × their u' attempted M0 if d/dx (tan 2x) =(2) $\sec^2 x$ isw
			OR $y = x^2 \dfrac{\sin 2x}{\cos 2x}$ $\dfrac{dy}{dx} = x^2 \dfrac{\cos 2x.2\cos 2x - \sin 2x(-2\sin 2x)}{\cos^2 2x} + 2x\dfrac{\sin 2x}{\cos 2x}$ = … = $2x^2\sec^2 2x + 2x\tan 2x$	M1 A1 A1cao	product rule correct expression or $2x^2/\cos^2 2x + 2x\tan 2x$ (isw)	*see additional notes for complete solution* u × their v' + v × their u' attempted or $(2x^2 + 2x\sin 2x\cos 2x)/\cos^2 2x$ or $2x^2/\cos^2 2x + 2x\sin 2x / \cos 2x$
			OR $y = \dfrac{x^2 \sin 2x}{\cos 2x}$ $\dfrac{dy}{dx} = \dfrac{\cos 2x(2x\sin 2x + x^2 2\cos 2x) - 2x^2 \sin 2x(-\sin 2x)}{\cos^2 2x}$ = … = $2x^2\sec^2 2x + 2x\tan 2x$	M1 A1 A1cao **[3]**	quotient rule correct expression or $2x^2/\cos^2 2x + 2x\tan 2x$ (isw)	*see additional notes for complete solution* (v × their u' − u × their v')/v^2 attempted or $(2x^2 + 2x\sin 2x\cos 2x)/\cos^2 2x$ or $2x^2/\cos^2 2x + 2x\sin 2x / \cos 2x$
2			fg(x) = $\ln(1+x^2)$ ($x \in \Re$) gf(x) = $1+(\ln x)^2$ ($x > 0$) $\ln(1+x^2)$ even $1 + (\ln x)^2$ neither	B1 B1 B1 B1 **[4]**	condone missing bracket, and missing or incorrect domains Penalise missing bracket Penalise missing bracket	If fg and gf the wrong way round, B1B0 not $1 + \ln(x^2)$
3			$u = x$, du/dx = 1, dv/dx = cos ½ x, v = 2sin ½ x $\int_0^{\pi/2} x\cos\frac{1}{2}x\,dx = \left[2x\sin\frac{1}{2}x\right]_0^{\pi/2} - \int_0^{\pi/2} 2\sin\frac{1}{2}x\,dx$ $= \left[2x\sin\frac{1}{2}x + 4\cos\frac{1}{2}x\right]_0^{\pi/2}$ $= \pi\sin\frac{\pi}{4} + 4\cos\frac{\pi}{4} - (2.0.\sin 0 + 4\cos 0)$ $= \pi.\frac{1}{\sqrt{2}} + 4.\frac{1}{\sqrt{2}} - 4$ $= \frac{\sqrt{2}}{2}\pi + 2\sqrt{2} - 4$*	M1 A1ft A1 M1 A1cao **[5]**	correct u, u', v, v' consistent with their u, v 2x sin ½ x + 4 cos ½ x oe (no ft) substituting correct limits into correct expression **NB AG**	but allow v to be any multiple of sin ½ x M0 if u = cos ½ x, , v' = x can be implied by one correct intermediate step

4			Cubes are 1, 8, 27, 64, 125, 216, 343, 512 [so false as] $8^3 = 512$	M1 A1 **[2]**	Attempt to find counter example counter-example identified (e.g. underlining, circling) [counter-examples all have 8 as units digit]	if no counter-example found, award M1 if at least 3 cubes are calculated. condone not explicitly stating statement is false
5	**(i)**		(One-way) stretch in y-direction, s.f. 2 or in x-direction s.f. ½ translation 1 to right (2 if followed by x-stretch) $y = 2\|x-1\|$	 B1 B1 B1 **[3]**	 must specify s.f. and direction o.e. e.g. $y = \|2x-2\|$ $y = \|2(x-1)\|$	Allow 'compress', 'squeeze'(for s.f. ½), but not 'enlarge', 'x-coordinates halved', etc Allow 'shift','move' or vector only, 'right 1' Don't allow misreads (e.g. transforming solid graph to dashed graph) Award B1 for one of these seen, and a second B1 if combined transformations are correct
5	**(ii)**		Reflection in x-axis or translation right $\pm\pi$ or rotation of 180° [about O] translation +1 in y-direction (− 1 if followed by reflection in x-axis $y = 1 - \cos x$	 B1 B1 B1 **[3]**	 $\begin{pmatrix}\pm\pi \\ 1\end{pmatrix}$ is B2 allow $1 + \cos(x \pm\pi)$ (bracket needed)	Translations as above. Reflection: must specify axis, allow 'flip' Rotation: condone no origin stated. *See additional notes for other possible solutions*. Award B1 for any one of these seen, and a second B1 if combined transformations are correct
6	**(i)**		When $t = 2$, $r = 20(1 - e^{-0.4}) = 6.59$ m $dr/dt = -20 \times (-0.2e^{-0.2t})$ $= 4e^{-0.2t}$ When $t = 2$, $dr/dt = 2.68$	M1A1 M1 A1 **[4]**	6.6 or art 6.59 $-0.2e^{-0.2t}$ soi 2.7 or art 2.68 or $4e^{-0.4}$	 mark final answer
6	**(ii)**		$A = \pi r^2$ $\Rightarrow \quad dA/dr = 2\pi r$ (= 41.428…) $dA/dt = (dA/dr) \times (dr/dt)$ $= 41.428\ldots \times 2.68$ $= 111$ m^2/hr	M1 A1 M1 A1 **[4]**	attempt to differentiate πr^2 $dA/dr = 2\pi r$ (not dA/dt, dr/dA etc) (o.e.) chain rule expressed in terms of their A, r or implied 110 or art 111	or differentiating $400\pi(1 - e^{-0.2t})^2$ M1 $dA/dt = 400\pi.2(1 - e^{-0.2t}).(-0.2e^{-0.2t})$ A1 substitute $t = 2$ into correct dA/dt M1 (Could use another letter for A)

7	(i)		$x^3 + y^3 = 3xy$ $\Rightarrow \quad 3x^2 + 3y^2(dy/dx) = 3x(dy/dx) + 3y$ $\Rightarrow \quad (3y^2 - 3x)(dy/dx) = 3y - 3x^2$ $\Rightarrow \quad dy/dx = (3y - 3x^2)/(3y^2 - 3x)$ $= (y - x^2)/(y^2 - x)$*	B1B1 M1 A1cao [4]	LHS, RHS Condone $3x\,dy/dx+y$ (i.e.with missing bracket) if recovered thereafter collecting terms in dy/dx and factorising NB **AG**	or equivalent if re-arranged. ft correct algebra on incorrect expressions with two dy/dx terms Ignore starting with '$dy/dx = \ldots$' unless pursued
7	(ii)		TP when $y - x^2 = 0$ $\Rightarrow \quad y = x^2$ $\Rightarrow \quad x^3 + x^6 = 3x.x^2$ $\Rightarrow \quad x^6 = 2x^3$ $\Rightarrow \quad x^3 = 2$ (or $x = 0$) $\Rightarrow \quad x = \sqrt[3]{2}$	 M1 M1 A1 A1cao [4]	 or $x = \sqrt{y}$ substituting for y in implicit eqn (allow one slip, e.g. x^5) o.e. (soi) must be exact	 or x for y (i.e. $y^{3/2} + y^3 = 3y^{1/2}y$ o.e.) or $y^{3/2} = 2$ $x = 1.2599\ldots$ is A0 (but can isw $x = \sqrt[3]{2}$)
8	(i)		When $x = 3$, $y = 3/\sqrt{(3 - 2)} = 3$ So P is (3, 3) which lies on $y = x$	M1 A1 **[2]**	substituting $x = 3$ (both x's) $y = 3$ and completion ('3 = 3' is enough)	or $x = x/\sqrt{(x - 2)}$ M1 $\Rightarrow x = 3$ A1(by solving or verifying)
8	(ii)		$\frac{dy}{dx} = \frac{\sqrt{x-2}.1 - x.\frac{1}{2}.(x-2)^{-1/2}}{x-2}$ $= \frac{x-2-\frac{1}{2}x}{(x-2)^{3/2}} = \frac{\frac{1}{2}x-2}{(x-2)^{3/2}}$ $= \frac{x-4}{2(x-2)^{3/2}}$* When $x = 3$, $dy/dx = -\tfrac{1}{2} \times 1^{3/2}$ $= -\tfrac{1}{2}$ This gradient would be -1 if curve were symmetrical about $y = x$	M1 A1 M1 A1 M1 A1 A1cao **[7]**	Quotient or product rule PR: $-\tfrac{1}{2}x(x - 2)^{-3/2} + (x - 2)^{-1/2}$ correct expression $\times$ top and bottom by $\sqrt{(x - 2)}$ o.e. e.g. taking out factor of $(x - 2)^{-3/2}$ NB **AG** substituting $x = 3$ or an equivalent valid argument	If correct formula stated, allow one error; otherwise QR must be on correct u and v, with numerator consistent with their derivatives and denominator correct initially allow ft on correct equivalent algebra from their incorrect expression

8	(iii)		$u = x - 2 \Rightarrow \mathrm{d}u/\mathrm{d}x = 1 \Rightarrow \mathrm{d}u = \mathrm{d}x$ When $x = 3$, $u = 1$ when $x = 11$, $u = 9$ $\Rightarrow \int_3^{11} \frac{x}{\sqrt{x-2}}\,\mathrm{d}x = \int_1^9 \frac{u+2}{u^{1/2}}\,\mathrm{d}u$ $= \int_1^9 (u^{1/2} + 2u^{-1/2})\,\mathrm{d}u$ $= \left[\frac{2}{3}u^{3/2} + 4u^{1/2}\right]_1^9$ $= (18 + 12) - (2/3 + 4)$ $= 25\frac{1}{3}$ * Area under $y = x$ is ½ (3 + 11) × 8 = 56 Area = (area under $y = x$) – (area under curve) so required area $= 56 - 25\frac{1}{3} = 30\frac{2}{3}$	B1 B1 M1 A1 M1 A1cao B1 M1 A1cao **[9]**	or $\mathrm{d}x/\mathrm{d}u = 1$ $\int \frac{u+2}{u^{1/2}}(\mathrm{d}u)$ splitting their fraction (correctly) and $u/u^{1/2} = u^{1/2}$ (or $\sqrt{u}$) $\left[\frac{2}{3}u^{3/2} + 4u^{1/2}\right]$ (o.e) substituting correct limits NB AG o.e. (e.g. 60.5 – 4.5) soi from working 30.7 or better	No credit for integrating initial integral by parts. Condone $\mathrm{d}u = 1$. Condone missing $\mathrm{d}u$'s in subsequent working. or integration by parts: $2u^{1/2}(u+2) - \int 2u^{1/2}\mathrm{d}u$ (must be fully correct – condone missing bracket by parts: $[2u^{1/2}(u+2) - 4u^{3/2}/3]$ F(9) – F(1) (u) or F(11) – F(3) (x) dep substitution and integration attempted must be trapezium area: $60.5 - 25\frac{1}{3}$ is M0
9	(i)		When $x = 1$, f(1) = ln(2/2) = ln 1 = 0 so P is (1, 0) f(2) = ln(4/3)	B1 B1 **[2]**	or $\ln(2x/1+x) = 0 \Rightarrow 2x/(1+x) = 1$ $\Rightarrow 2x = 1+x \Rightarrow x = 1$	if approximated, can isw after ln(4/3)
9	(ii)		$y = \ln(2x) - \ln(1 + x)$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x} - \frac{1}{1+x}$ **OR** $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x}{1+x}\right) = \frac{(1+x)2 - 2x.1}{(1+x)^2} = \frac{2}{(1+x)^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{(1+x)^2}.\frac{1}{2x/(1+x)} = \frac{1}{x(1+x)}$ At P, $\mathrm{d}y/\mathrm{d}x = 1 - ½ = ½$	M1 M1 A1cao B1 M1 A1 A1cao **[4]**	one term correct mark final ans correct quotient or product rule chain rule attempted o.e., but mark final ans	condone lack of brackets $2/2x$ or $-1/(1+x)$ need not be simplified need not be simplified

9	(iii)		$x = \ln[2y/(1+y)]$ or $\Rightarrow e^x = 2y/(1+y)$ $\Rightarrow e^x(1+y) = 2y$ $\Rightarrow e^x = 2y - e^x y = y(2 - e^x)$ $\Rightarrow y = e^x/(2 - e^x)$ [= g(x)] **OR** $gf(x) = g(2x/(1+x)) = e^{\ln[2x/(1+x)]}/\{2-e^{\ln[2x/(1+x)]}\}$ $= \dfrac{2x/(1+x)}{2-2x/(1+x)}$ $= \dfrac{2x}{2+2x-2x} = \dfrac{2x}{2} = x$ gradient at R = 1/ ½ = 2	 B1 B1 B1 B1 M1 A1 M1A1 B1 ft **[5]**	($x \leftrightarrow y$ here or at end to complete) completion forming gf or fg 1/their ans in (ii) unless ±1 or 0	$x = e^y/(2 - e^y)$ $x(2 - e^y) = e^y$ B1 $2x = e^y + xe^y = e^y(1 + x)$ B1 $2x/(1+x) = e^y$ B1 $\ln[2x/(1+x)] = y$ [= f(x)] B1 fg(x) $= \ln\{2e^x/(2-e^x)/[1+ e^x/(2-e^x)]\}$ M1 $= \ln[2e^x/(2 - e^x + e^x)]$ A1 $= \ln(e^x) = x$ M1A1 2 must follow ½ for 9(ii) unless g′(x) used (*see additional notes*)
9	(iv)		let $u = 2 - e^x \Rightarrow du/dx = -e^x$ $x = 0$, $u = 1$, $x = \ln(4/3)$, $u = 2 - 4/3 = 2/3$ $\Rightarrow \int_0^{\ln(4/3)} g(x)\,dx = \int_1^{2/3} -\frac{1}{u}\,du$ $= \left[-\ln(u)\right]_1^{2/3} = -\ln(2/3) + \ln 1 = \ln(3/2)$* Shaded region = rectangle – integral = 2ln(4/3) – ln(3/2) = ln(16/9 × 2/3) = ln(32/27)*	 B1 M1 A1 A1cao M1 B1 A1cao **[7]**	$2-e^0 = 1$, and $2 - e^{\ln(4/3)} = 2/3$ seen $\int -1/u\,du$ condone $\int 1/u\,du$ [–ln(u)] (could be [ln$u$] if limits swapped) NB AG rectangle area = 2ln(4/3) NB AG must show at least one step from 2ln(4/3) – ln(3/2)	here or later (i.e. after substituting 0 and ln(4/3) into $\ln(2 - e^x)$) or by inspection [$k \ln(2 - e^x)$] $k = -1$ Allow full marks here for correctly evaluating $\int_1^2 \ln(\frac{2x}{1+x})\,dx$ (*see additional notes*)

Additional notes and solutions

1. $y = x^2 \dfrac{\sin 2x}{\cos 2x}$ $\quad \dfrac{dy}{dx} = x^2 \dfrac{\cos 2x.2\cos 2x - \sin 2x(-2\sin 2x)}{\cos^2 2x} + 2x\dfrac{\sin 2x}{\cos 2x} = x^2 \dfrac{2\cos^2 2x + 2\sin^2 2x}{\cos^2 2x} + 2x\dfrac{\sin 2x}{\cos 2x}$

$= x^2 \dfrac{2}{\cos^2 2x} + 2x\dfrac{\sin 2x}{\cos 2x} = 2x^2 \sec^2 2x + 2x\tan 2x$

$y = \dfrac{x^2 \sin 2x}{\cos 2x}$ $\quad \dfrac{dy}{dx} = \dfrac{\cos 2x(2x\sin 2x + x^2 2\cos 2x) - 2x^2 \sin 2x(-\sin 2x)}{\cos^2 2x}$

$= \dfrac{2x\cos 2x \sin 2x + 2x^2 \cos^2 2x - x^2 \sin 2x(-2\sin^2 2x)}{\cos^2 2x} = \dfrac{2x\cos 2x\sin 2x + 2x^2\cos^2 2x + 2x^2 \sin^2 2x}{\cos^2 2x}$

$= \dfrac{2x\cos 2x \sin 2x + 2x^2}{\cos^2 2x} = 2x\tan 2x + 2x^2 \sec^2 x$

5 (ii) translation $\begin{pmatrix}0\\1\end{pmatrix}$ then translation $\begin{pmatrix}\pm\pi\\0\end{pmatrix}$ translation $\begin{pmatrix}\pm\pi\\0\end{pmatrix}$ then translation $\begin{pmatrix}0\\1\end{pmatrix}$ translation $\begin{pmatrix}\pm\pi\\1\end{pmatrix}$ B2

translation $\begin{pmatrix}0\\1\end{pmatrix}$ then reflection in $y = 1$ translation $\begin{pmatrix}0\\1\end{pmatrix}$ then reflection in $x = ½ \pi$ translation $\begin{pmatrix}0\\-1\end{pmatrix}$ then reflection in x-axis

reflection in x-axis then translation $\begin{pmatrix}0\\1\end{pmatrix}$ reflection in $y = 1$ then translation $\begin{pmatrix}0\\-1\end{pmatrix}$ rotation 180° about O then translation $\begin{pmatrix}0\\1\end{pmatrix}$

reflection in $y = ½$ B2

9(iii) last part: $g(x) = e^x/(2 - e^x) \Rightarrow g'(x) = [(2 - e^x)e^x - e^x(-e^x)]/(2 - e^x)^2 = 2e^x/(2 - e^x)^2$
or $g'(x) = e^x(-1)(-e^x)](2 - e^x)^{-2} + e^x(2 - e^x)^{-1}$
$g'(0) = 2.1/1^2 = 2$ B1

9(iv) last part

$\int_1^2 \ln(\dfrac{2x}{1+x})\,dx = \int_1^2 (\ln 2 + \ln x - \ln(1+x))\,dx \quad = \left[x\ln 2 + x\ln x - x - (1+x)\ln(1+x) + x\right]_1^2$

$= 2\ln 2 + 2\ln 2 - 2 - 3\ln 3 + 2 - (\ln 2 - 1 - 2\ln 2 + 1) = 5\ln 2 - 3\ln 3 = \ln(32/27)$