MEI

Mathematics in Education and Industry

MEI STRUCTURED MATHEMATICS

METHODS FOR ADVANCED MATHEMATICS, C3

Practice Paper C3-A

Additional materials: Answer booklet/paper
Graph paper
List of formulae (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS

- Write your Name on each sheet of paper used or the front of the booklet used.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION

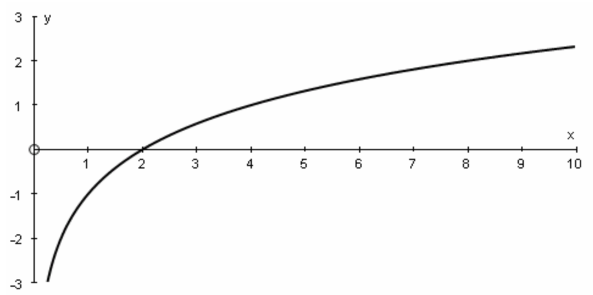
- The number of marks is given in brackets [] at the end of each question or part-question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is **72.**
- **You are reminded of the need for clear presentation in your answers.**

Section A (36 marks)

1 Prove that the product of consecutive integers is always even. [2]

2 Find $\dfrac{dy}{dx}$ when $y = \sqrt{1 + x^3}$. [3]

3 The graph shows part of the function $y = a\ln(bx)$.



The graph passes through the points (2, 0) and (4, 1).

(i) Show that $b = \dfrac{1}{2}$ and find the exact value of a. [3]

(ii) Solve the inequality $|a\ln(bx)| < 2$. [4]

4 **(i)** Show that $y = axe^{-x}$ for $a > 0$ has only one stationary point for all values of x. Determine whether this stationary value is a maximum or minimum point. [5]

(ii) Sketch the curve. [2]

5 Find $\int_2^3 xe^{2x}\,dx$, giving your answer to 1 decimal place. [5]

6 Find $\dfrac{d}{dx}(x\ln x)$ and hence or otherwise find the value of $\int_2^3 \ln x\,dx$, giving your answer in the form $\ln a + b$, where a and b are to be determined. [6]

7 Two quantities, x and θ, vary with time and are related by the equation $x = 5\sin\theta - 4\cos\theta$.

(i) Find the value of x when $\theta = \dfrac{\pi}{2}$. [1]

(ii) When $\theta = \dfrac{\pi}{2}$, its rate of increase (in suitable units) is given by $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 0.1$.

Show that at that moment $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0.4$. [5]

Section B (36 marks)

8 You are given that $\mathrm{f}(x) = \dfrac{x}{x^2+1}$ for all real values of x.

(i) Show that $\mathrm{f}'(x) = \dfrac{1-x^2}{\left(x^2+1\right)^2}$. [3]

(ii) Hence show that there is a stationary value at $\left(1, \dfrac{1}{2}\right)$ and find the coordinates of the other stationary point. [2]

(iii) The graph of the curve is shown in Fig. 8.

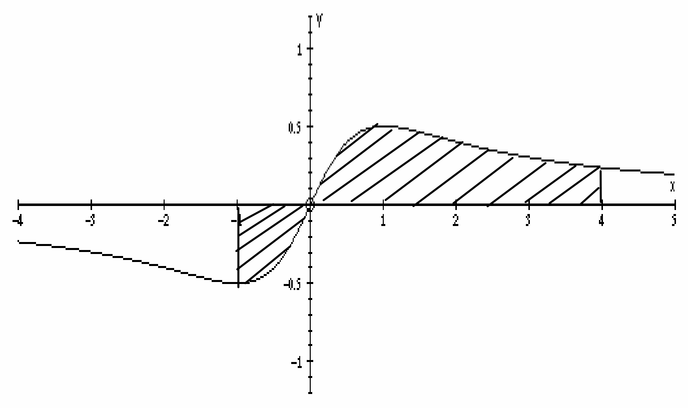


Fig. 8

State whether the curve is odd or even and prove the result algebraically. [2]

(iv) Show that $\displaystyle\int_1^4 \frac{x}{x^2+1}\,\mathrm{d}x = \int_a^b k\frac{1}{u+1}\,\mathrm{d}u$, where the values of a, b and k are to be determined.

[5]

(v) Hence find the area of the shaded region in Fig. 8. [6]

9 The curve in Fig. 9.1 has equation $\sqrt{x}+\sqrt{y}=1$.

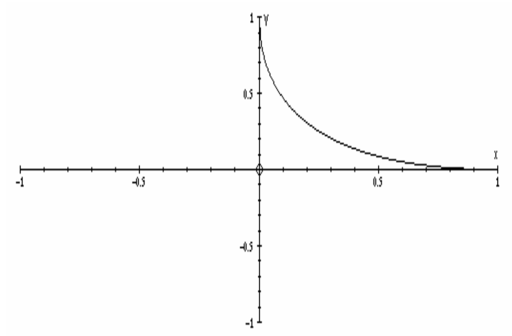


Fig. 9.1

(i) Show that this is part, but not all of the curve $y=1-2\sqrt{x}+x$.

Sketch the full curve $y=1-2\sqrt{x}+x$. [7]

(ii) Fig.9.2 shows a star shape made up of four parts, one of which is given in part (i) above.

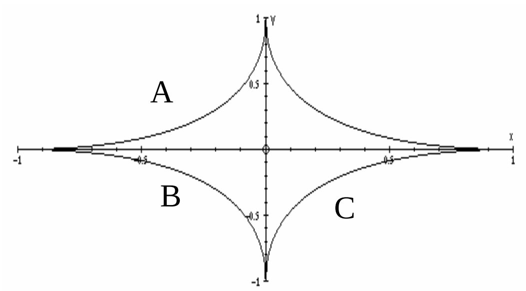


Fig. 9.2

For each of the sections of the shape labelled A, B and C, state the equation of the curve and the domain. [6]

(iii) The shape shown in Fig.9.2 is made into that in Fig. 10.3 by stretching the part of the figure for which $y > 0$ by a scale factor of 2.

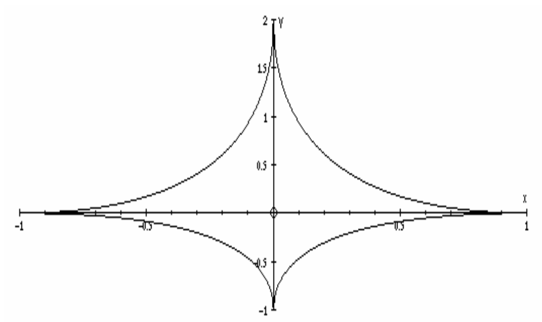


Fig. 9.3

Find the area of this shape. [5]

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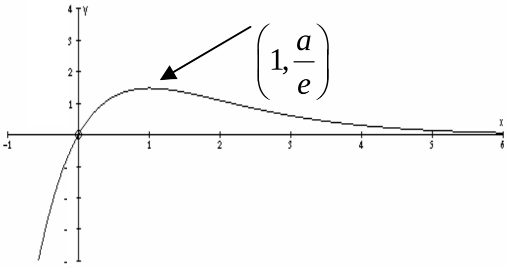
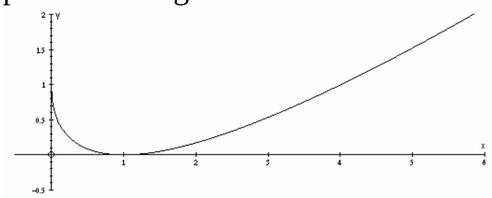
MARK SCHEME

Qu		Answer	Mark	Comment
Section A				
1		Product of two numbers, one of which is even is always even. Two consecutive numbers contain an even number. *OR* acceptable alternatives	B1 B1 **2**	
2		$y=\sqrt{1+x^3}$ Let $u=1+x^3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=3x^2$ $y=u^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{2}u^{-\frac{1}{2}}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}u^{-\frac{1}{2}}\times 3x^2=\frac{3}{2}\frac{x^2}{\sqrt{1+x^3}}$	M1 A1 A1 **3**	Chain rule $\frac{\mathrm{d}y}{\mathrm{d}u}$ Answer
3	**(i)**	Substitute: $0 = a\ln 2b \Rightarrow \ln 2b = 0 \Rightarrow 2b = 1 \Rightarrow b = ½$ $1 = a\ln 2 \Rightarrow a = \frac{1}{\ln 2}$	M1 A1 A1 **3**	
	(ii)	$\|a\ln(bx)\|<2 \Rightarrow \left\|\frac{\ln\frac{1}{2}x}{\ln 2}\right\|<2 \Rightarrow \left\|\ln\frac{1}{2}x\right\|<2\ln 2$ $\Rightarrow -2\ln 2<\ln\frac{1}{2}x<2\ln 2$ $\Rightarrow \ln\frac{1}{4}<\ln\frac{1}{2}x<\ln 4$ $\Rightarrow \frac{1}{4}<\frac{1}{2}x<4 \Rightarrow \frac{1}{2}<x<8$	 M1 M1 A1 A1 **4**	 Modulus Powers of logs
4	**(i)**	$y=ax\mathrm{e}^{-x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=a\mathrm{e}^{-x}-ax\mathrm{e}^{-x}=a\mathrm{e}^{-x}(1-x)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow x=1$ only at $\left(1,\frac{a}{e}\right)$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-a\mathrm{e}^{-x}(1-x)-a\mathrm{e}^{-x}$: When $x=1, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}<0$ $\Rightarrow$ Maximum	M1 A1 M1 A1 B1 **5**	Product = 0 or any equivalent argument
	(ii)	 	B1 B1 **2**	For curve for stationary point

5		$\int_2^3 xe^{2x}\,dx \qquad u = x,\ \ \frac{dv}{dx} = e^{2x}$	M1	Choice of u
		$\frac{du}{dx} = 1,\ \ v = \frac{1}{2}e^{2x}$	A1	
		$= \left[\frac{1}{2}x\,e^{2x}\right]_2^3 - \frac{1}{2}\int_2^3 e^{2x}\,dx = \left[\frac{1}{2}x\,e^{2x} - \frac{1}{4}e^{2x}\right]_2^3$	M1 A1	
		$= \frac{5}{4}e^6 - \frac{3}{4}e^4 = 463.3$	A1 **5**	
6		$\frac{d}{dx}(x\ln x) = \ln x + x \times \frac{1}{x} = \ln x + 1$	M1 A1	Product
		$\Rightarrow x\ln x = \int(\ln x + 1)\,dx = \int \ln x\,dx + x$	M1	
		$\Rightarrow \int_2^3 \ln x\,dx = [x\ln x - x]_2^3 = (3\ln 3 - 3) - (2\ln 2 - 2)$ $= 3\ln 3 - 2\ln 2 - 1$ $= \ln\frac{27}{4} - 1$	A1 M1 A1 **6**	Integrand limits
7	**(i)**	$x = 5\sin\frac{\pi}{2} - 4\cos\frac{\pi}{2} = 5$	B1 **1**	
	(ii)	$\frac{dx}{d\theta} = 5\cos\theta + 4\sin\theta$: When $\theta = \frac{\pi}{2},\ \frac{dx}{d\theta} = 4$ $\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = 4 \times 0.1 = 0.4$	M1 A1 A1 M1 E1 **5**	4
Section B				
8	**(i)**	$f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{(x^2+1).1 - x.2x}{(x^2+1)^2}$	M1 A1	Formula Middle section
		$= \frac{1-x^2}{(x^2+1)^2}$	E1 **3**	answer
	(ii)	$f'(x) = \frac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$		
		When $x = 1$, $f(x) = \frac{1}{1+1} = \frac{1}{2}$ i.e. $\left(1,\ \frac{1}{2}\right)$	E1	Substitute Find $f(x)$
		Other stationary point is when $x = -1$, $f(x) = \frac{-1}{1+1} = -\frac{1}{2}$ i.e. $\left(-1,\ -\frac{1}{2}\right)$	B1 **2**	
	(iii)	The graph is odd. $f(-x) = \frac{-x}{(-x)^2+1} = -\frac{x}{x^2+1} = -f(x)$	B1 B1 **2**	

	(iv)	$u = x^2 \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$ When $x = 1, u = 1$ When $x = 4,\ u = 16$ $\Rightarrow \int_1^4 \frac{x}{x^2+1}\mathrm{d}x = \int_1^{16} \frac{1}{u+1}.\frac{1}{2}\mathrm{d}u$ $\Rightarrow a = 1, b = 16, k = \frac{1}{2}$	M1 B1 B1 A1 A1 **5**	
	(v)	Because the function is odd the area in [-1,0] is equal in magnitude but opposite in sign to the area in [0,1] So shaded area = $\int_1^4 \frac{x}{x^2+1}\mathrm{d}x + 2\int_0^1 \frac{x}{x^2+1}\mathrm{d}x$ $= \int_1^{16} \frac{1}{u+1}.\frac{1}{2}\mathrm{d}u + 2\int_0^1 \frac{1}{u+1}.\frac{1}{2}\mathrm{d}u$ $= \frac{1}{2}\left[\ln(u+1)\right]_1^{16} + 2\frac{1}{2}\left[\ln(u+1)\right]_0^1$ $= \frac{1}{2}\ln\frac{17}{2} + 2\frac{1}{2}\ln 2 = \frac{1}{2}\left(\ln\frac{17}{2} + \ln 4\right) = \frac{1}{2}\ln 34 \approx 1.763$	B1 M1 A1 M1 A1 A1 **6**	$\ln(u + 1)$
9	**(i)**	$\sqrt{y} = 1 - \sqrt{x}$ $\Rightarrow y = \left(1-\sqrt{x}\right)^2 = 1 + x - 2\sqrt{x}$ This is undefined for $x < 0$ but is defined for $x > 1$ so that part is missing. 	M1 A1 A1 B1 B1 B1 B1 **7**	Or: squaring introduces also the negative arm $x < 0$ $x > 0$ Extra range shape
	(ii)	A is $\sqrt{-x} + \sqrt{y} = 1$ for $-1 \le x \le 0$ B is $\sqrt{-x} + \sqrt{-y} = 1$ for $-1 \le x \le 0$ C is $\sqrt{x} + \sqrt{-y} = 1$ for $0 \le x \le 1$	B1 B1 B1 B1 B1 B1 **6**	
	(iii)	Area in Fig. 9.1 is $\int_0^1 \left(1 + x - 2\sqrt{x}\right)\mathrm{d}x$ $= \left[x + \frac{1}{2}x^2 - \frac{4}{3}x^{\frac{3}{2}}\right]_0^1 = 1 + \frac{1}{2} - \frac{4}{3} = \frac{1}{6}$ $\Rightarrow$ Area of shape is $\frac{1}{6} + \frac{1}{6} + 2\times\frac{1}{6} + 2\times\frac{1}{6} = 1$	M1 A1 A1 M1 A1 **5**	