MEI

Mathematics in Education and Industry

MEI STRUCTURED MATHEMATICS

METHODS FOR ADVANCED MATHEMATICS, C3

Practice Paper C3-C

Additional materials: Answer booklet/paper
Graph paper
List of formulae (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS

- Write your Name on each sheet of paper used or the front of the booklet used.
- There is an Insert booklet for use in Questions **3** and **9.**
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION

- The number of marks is given in brackets [] at the end of each question or part-question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is **72.**
- **You are reminded of the need for clear presentation in your answers.**

Section A (36 marks)

1 John asserts that the expression $n^2 + n + 11$ is prime for all positive integer values of n. Show that John is wrong in his assertion. [2]

2 **(i)** Show that $f(x) = |x^3|$ is an even function. [2]

(ii) It is suggested that the function $g(x) = (x-1)^3$ is odd. Prove that this is false. [2]

3 **Answer this question on the Insert provided.**

The function f(x) is a periodic function which is defined for all values of x.
Fig. 3 shows the graph shows the function $y = f(x)$ for $-4 \le x \le 4$.

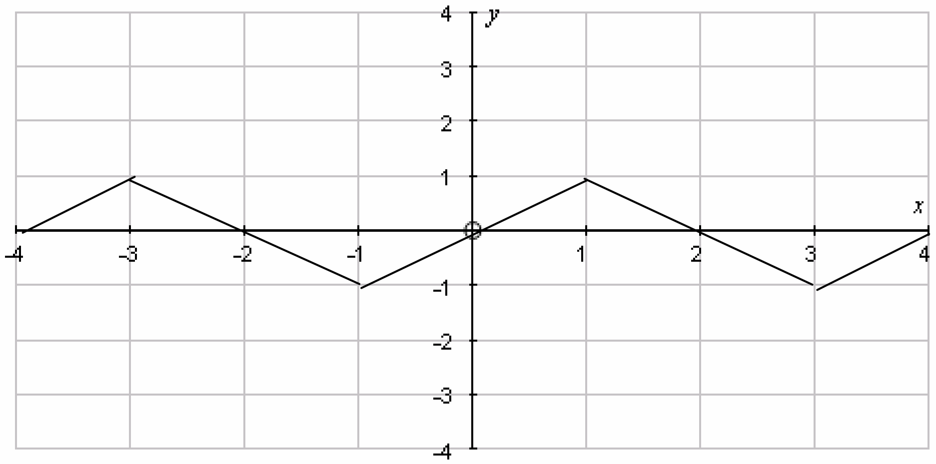

Fig. 3

Use the Insert to sketch separately the graphs of the following functions:

(i) $y = 2f(x)$, **(ii)** $y = f(2x)$. [4]

4 The volume of a sphere, V cm^3 is given by the formula $V = \frac{4}{3}\pi r^3$ where r cm is the radius.

The radius of a sphere increases at a constant rate of 2 cm per second.

Find the rate of increase of V when $r = 10$ cm. [5]

5 The equation of a circle is $x^2 + y^2 = 25$.

(i) Show that $\dfrac{dy}{dx} = -\dfrac{x}{y}$. [3]

(ii) Hence find the equation of the normal to the circle at the point (3, 4). [3]

6 **(a)** Find $\int x \cos 2x \, dx$. [3]

(b) Using the substitution $u = x^2 + 1$, or otherwise, find the exact value of $\int_2^3 \dfrac{x}{x^2+1} dx$. [5]

7 Fig. 7 shows the graphs of the curves $y = e^{-x}$ and $y = e^{-x} \sin x$ for $0 \le x \le \pi$.

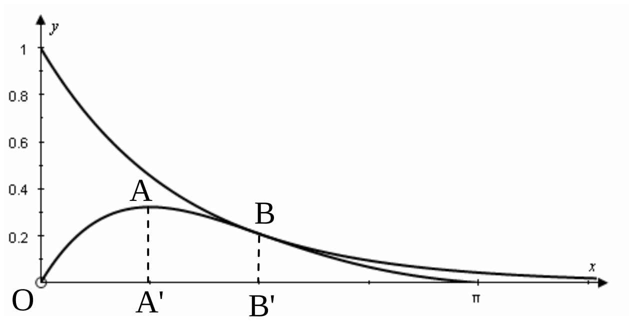

Fig. 7

The maximum point on $y = e^{-x} \sin x$ is at A, and the curves touch at B.
A' and B' are the points on the x-axis such that A' A and B' B are parallel to the y-axis.

Show that OA' = A' B'. [7]

Section B (36 marks)

8 Fig. 8 shows part of the graph of the function $y = 5x(2x - 1)^3$.

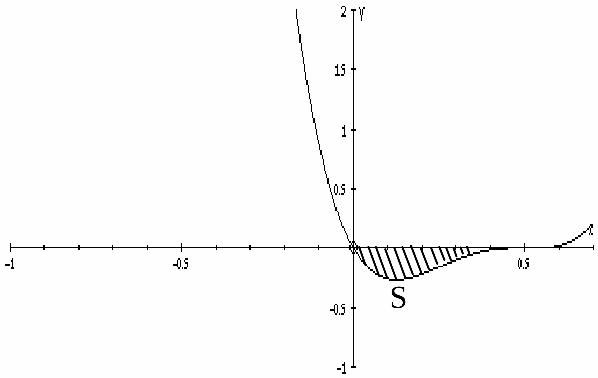

Fig. 8

(i) Find $\dfrac{dy}{dx}$ and hence find the x-coordinate of S, the turning point of the curve. [4]

(ii) Find the area of the shaded region enclosed between the curve and the x-axis. [6]

(iii) Given that $f(x) = 5x(2x - 1)^3$, show that $f(x + 0.5) = 40x^3(x + 0.5)$. [2]

(iv) Find $\displaystyle\int_{-\frac{1}{2}}^{0} 40x^3(x + 0.5)\,dx$. [4]

(v) Explain, with the aid of a sketch, the connection between your answer to parts (ii) and (iv). [2]

9 Answer parts (ii) and (iii) of this question on the Insert provided.

The bat population of a colony is being investigated and data are collected of the estimated number of bats in the colony at the beginning of each year.

It is thought that the population may be modelled by the formula

$$P = P_0 e^{kt}$$

where P_0 and k are constants, P is the number of bats and t is the number of years after the start of the collection of data.

(i) Explain why a graph of $\ln P$ against t should give a straight line. State the gradient and intercept of this line. [3]

(ii) The data collected are as follows.

Time (t years)	0	1	2	3	4
Number of bats, P	100	170	300	340	360

Using the first three pairs of data in the table, plot $\ln P$ against t on the axes given on the Insert, and hence estimate values for P_0 and k.
(Work to three significant figures.) [6]

This model assumes exponential growth, and assumes that once born a bat does not die, continuing to reproduce. This is unrealistic and so a second model is proposed with formula

$$P = 150\arctan(t-1) + 170 .$$

(*You are reminded that arctan values should be given in radians*.)

(iii) Plot on a single graph on the Insert the curves $P = P_0 e^{kt}$ for your values of P_0 and k and $P = 150\arctan(t-1) + 170$. The data pairs in the table above have been plotted for you. [5]

(iv) Using the second model calculate an estimate of the number of years it is before the bat population exceeds 375. [4]

NAME:………………………………………

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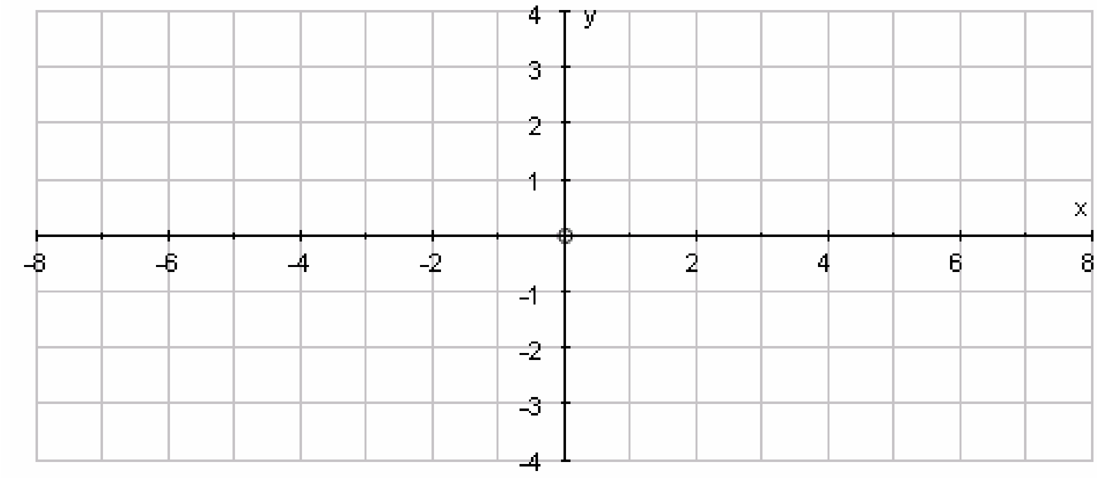
Practice Paper C3-C
INSERT

INSTRUCTIONS

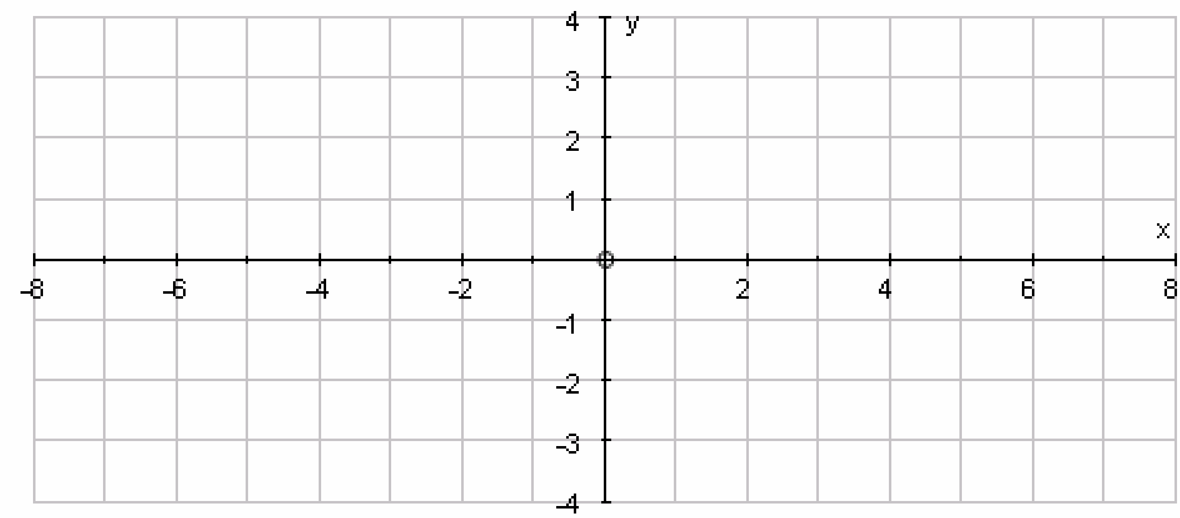
- This insert should be used for questions **3** and **9.**
- Write your name in the space at the top of this sheet.
- Attach this insert to the rest of your answers.

Insert for question 3.

(i) Sketch the graph of $y = 2\mathrm{f}(x)$

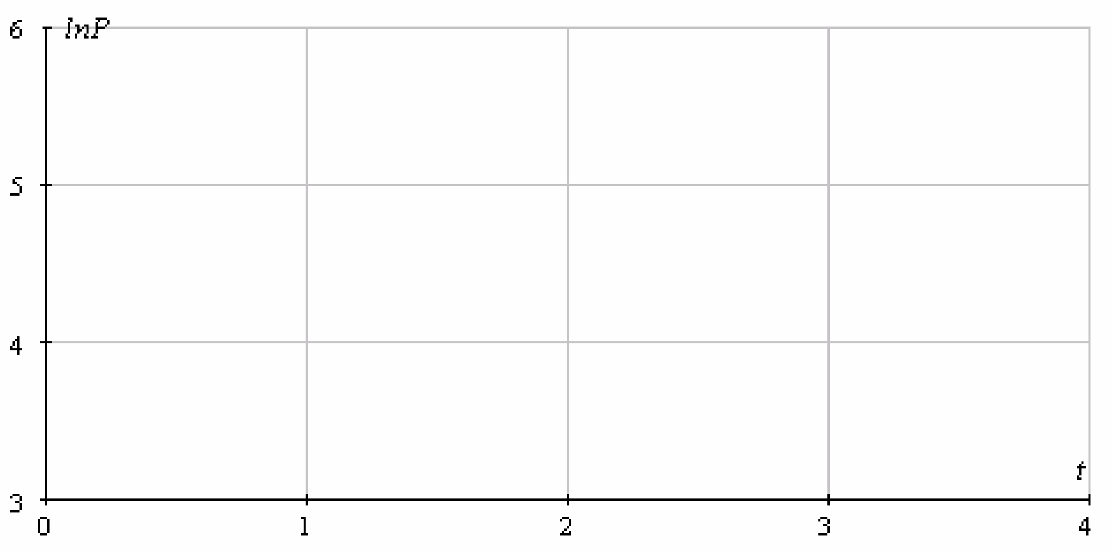


(ii) Sketch the graph of $y = \mathrm{f}(2x)$.

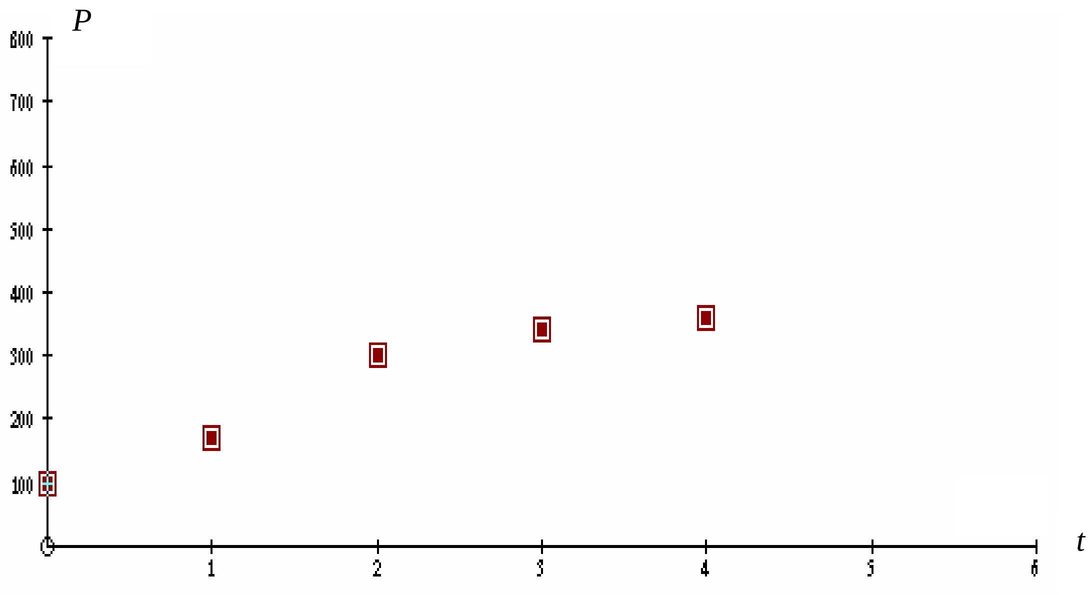


Insert for question 9.

(ii) Plot $\ln P$ against t.



(iii) Plot the curves $P = P_0e^{kt}$ and $P = 150\arctan(t - 1) + 170$ for your values of P_0 and k. The data pairs are plotted on the graph.



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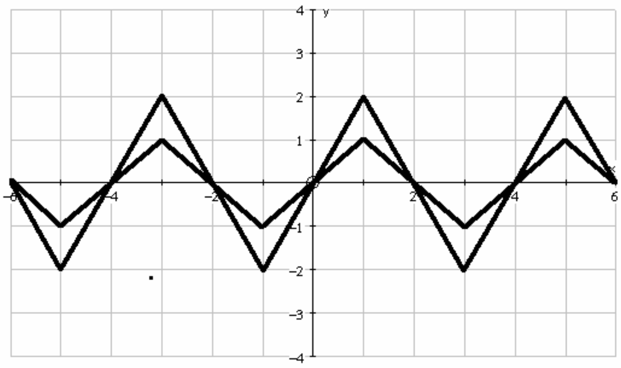
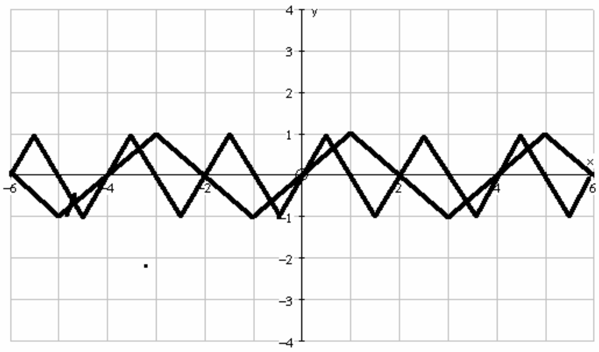
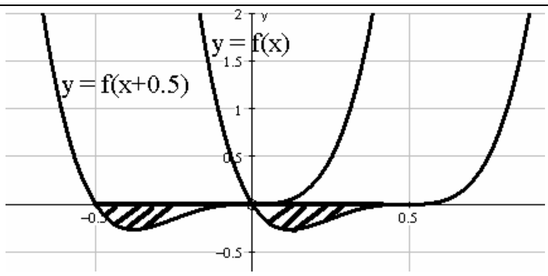
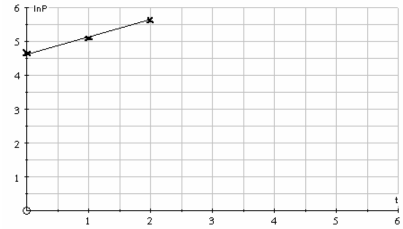
MARK SCHEME

Qu		Answer	Mark	Comment
Section A				
1		Take $n = 11$; the result is divisible by 11. ($n = 10$ is the smallest number)	M1A1 **2**	
2	**(i)**	$f(-x)$ $= \lvert(-x)^3\rvert$ $= \lvert -x^3\rvert$ $= \lvert x^3\rvert$ $= f(x)$ This argument holds regardless of whether x is positive, negative or zero.	M1 A1 **2**	Using $-x$
	(ii)	$g(-x)$ $= (-x-1)^3$ $g(1) \neq g(-1)$ (say), so the function is not odd.	M1 A1 **2**	
3	**(i)**		B1 B1 **2**	One for the correct amplitude and one for a function of the right period correctly placed.
	(ii)		B1 B1 **2**	One for a function of the right period and one for the correct amplitude correctly placed.
4		$V = \frac{4}{3}\pi r^3$ $\Rightarrow \frac{dV}{dr} = 4\pi r^2$ and $\frac{dr}{dt} = 2$ $\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt}$ $= 4\pi \times 10^2 \times 2$ $= 800\pi$ $(= 2513.27...)$ The rate of increase of V is 2500 cm ^{3}/sec, to 2sf (say).	M1 B1 B1 A1 A1 **5**	B1 for number, B1 for units.

5	**(i)**	$x^2+y^2=25$ $\Rightarrow 2x+2y\frac{dy}{dx}=0$ $\Rightarrow \frac{dy}{dx}=-\frac{x}{y}$	M1A1 E1 **3**	
	(ii)	Gradient of normal = $\frac{4}{3}$ $\Rightarrow y-4=\frac{4}{3}(x-3)$ $\Rightarrow 3y=4x$ Or equivalent.	B1 M1 A1 **3**	
6	**(a)**	$\int x\cos 2x dx$ $=\frac{x}{2}\sin 2x-\int\frac{1}{2}\sin 2x dx$ $=\frac{x}{2}\sin 2x+\frac{1}{4}\cos 2x+c$	M1 A1 A1 **3**	
	(b)	$\int_2^3 \frac{x}{x^2+1}dx$ $u=x^2+1\Rightarrow\frac{du}{dx}=2x$ $x=2\Rightarrow u=5,\ x=3\Rightarrow u=10$ $=\frac{1}{2}\int_5^{10}\frac{1}{u}du=\frac{1}{2}[\ln u]_5^{10}=\frac{1}{2}\ln 2$	M1 A1 A1 A1 A1 **5**	Substitution Integral in u Limits Integral Answer
7		For turning point $y=e^{-x}\sin x$ $\Rightarrow\frac{dy}{dx}=-e^{-x}\sin x+e^{-x}\cos x$ $=0$ when $\sin x=\cos x\Rightarrow x=\frac{\pi}{4}$ (in this range) $\Rightarrow$ OA' $=\frac{\pi}{4}$ For intersection: $e^{-x}\sin x=e^{-x}\Rightarrow\sin x=1$ $\Rightarrow x=\frac{\pi}{2}$(in this range) $\Rightarrow$ OB' $=\frac{\pi}{2}\Rightarrow$ A'B' $=\frac{\pi}{4}$	M1 A1 M1 A1 M1 A1 A1 **7**	Product = 0 Solving

Section B				
8	(i)	$\frac{dy}{dx} = 5x.6(2x-1)^2 + 5(2x-1)^3$ $= 5(2x-1)^2(8x-1)$ Where the gradient is zero, $x = \frac{1}{2}$ (double root, where the curve touches the x-axis) or $x = \frac{1}{8}$ (at S).	M1A1 M1 A1 **4**	
	(ii)	Area has magnitude $\left\lvert \int_0^{\frac{1}{2}} 5x(2x-1)^3\,dx \right\rvert$ $= \left\lvert \int_0^{\frac{1}{2}} (40x^4 - 60x^3 + 30x^2 - 5x)\,dx \right\rvert$ $= \left\lvert \left[8x^5 - 15x^4 + 10x^3 - \frac{5x^2}{2} \right]_0^{\frac{1}{2}} \right\rvert$ $= \left\lvert \frac{1}{4} - \frac{15}{16} + \frac{10}{8} - \frac{5}{8} \right\rvert$ $= \left\lvert -\frac{1}{16} \right\rvert$ So area is $\frac{1}{16}$ units^2. .	B1 M1A1 A2 A1 **6**	(or at end if the -ve sign is explained) Expand or integrate by parts A1 for some terms right *Alternatively by substitution*
	(iii)	$f(x+0.5)$ $= 5(x+0.5)(2[x+0.5]-1)^3$ $= 5(x+0.5)(2x+1-1)^3$ $= 5(x+0.5) \times 8x^3$ $= 40x^3(x+0.5)$	M1 M1 **2**	Correct step (ans given)
	(iv)	$\int_{-\frac{1}{2}}^{0} 40x^3(x+0.5)\,dx$ $= \int_{-\frac{1}{2}}^{0} (40x^4 + 20x^3)\,dx$ $= \left[8x^5 + 5x^4 \right]_{-\frac{1}{2}}^{0} = 0 - \left(-\frac{8}{32} + \frac{5}{16} \right) = -\frac{1}{16}$	M1 A1 A1 A1 **4**	Multiply out Terms Both terms
	(v)	 The area representing the one integral is a translation of that representing the other, so their values are equal.	B1 E1 **2**	

<table>
<tr>
<td>9</td>
<td>(i)</td>
<td>$P = P_0 e^{kt}$
$\Rightarrow \ln P = \ln P_0 + kt$
So if y is identified with lnP, m with k and x with t, we have $y = mx + c$; gradient k, intercept lnP_0.</td>
<td>M1A1

E1
3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(ii)</td>
<td>
<table>
<tr><td>T</td><td>P</td><td>ln P (to 3dp)</td></tr>
<tr><td>0</td><td>100</td><td>4.61</td></tr>
<tr><td>1</td><td>170</td><td>5.14</td></tr>
<tr><td>2</td><td>300</td><td>5.70</td></tr>
</table>

Gradient estimate (about)
$k = 0.55$;
Intercept (about) 4.61;
So $P_0 = \exp(4.61) = 100$,
to 2 sf.
</td>
<td>B1

B1
B1

M1
A1
B1
6</td>
<td>for table of values
points plotted
Straight line

Gradient

P_0</td>
</tr>
<tr>
<td></td>
<td>(iii)</td>
<td>
<table>
<tr><td>t</td><td>P (to 1 dp)</td></tr>
<tr><td>0</td><td>52.2</td></tr>
<tr><td>1</td><td>170</td></tr>
<tr><td>2</td><td>287.8</td></tr>
<tr><td>3</td><td>336.1</td></tr>
<tr><td>4</td><td>357.4</td></tr>
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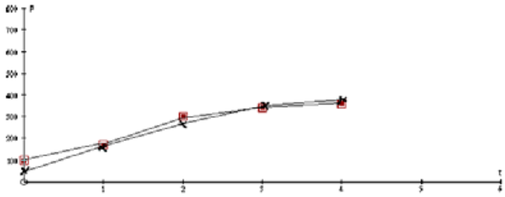
</td>
<td>M1
A1

M1
A1
5</td>
<td>For 1st curve

For 2nd curve</td>
</tr>
<tr>
<td></td>
<td>(iv)</td>
<td>$375 = 150\arctan(t-1) + 170$
$\Rightarrow t = 1 + \tan\frac{205}{150}$
$= 5.83....$
So the population will exceed 375 in 6 years.</td>
<td>B1

M1
A1

B1
4</td>
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