QUESTION 2

If $p(x)$ and $q(x)$ are polynomials of degree m and n respectively then $p(q(x))$ is a polynomial of degree $m \times n$.

Part (i)

Let $p(x)$ be a polynomial of degree $n \in \mathbb{N}$.

Then $p(p(p(x)))$ is a polynomial of degree n^3.

Further $p(p(p(x))) - 3p(x)$ will also be a polynomial of degree n^3.

Since this is equal to a polynomial of degree 1 then we must have that $n^3 = 1 \Rightarrow n = 1$.

Therefore let $p(x) = ax + b \Rightarrow p(p(p(x))) = a(a(ax+b)+b)+b$

We have $a(a(ax+b)+b)+b-3(ax+b) = -2x$ for all x.

$$\Rightarrow (a^3 - 3a)x + (a^2b + ab - 2b) = -2x$$

$$\Rightarrow a^3 - 3a = -2 \Rightarrow a^3 - 3a + 2 = 0 \Rightarrow (a-1)^2(a+2) = 0$$

So that $a = 1$ or -2.

We also have $a^2b + ab = 2b = 0$

Now $a = 1$ and $a = -2$ both allow b to have any value.

Therefore only polynomials that can satisfy this equation are $p(x) = x + b$ and $p(x) = -2x + b$ for an arbitrary constant b.

Part (ii)

Clearly the most general polynomial that can satisfy this equation has degree 2.

Let $p(x) = ax^2 + bx + c$ for $a, b, c \in \mathbb{R}$

Now $\left[p(x)\right]^2 = a^2x^4 + 2abx^3 + \left(2ac + b^2\right)x^2 + 2bcx + c^2$

Further $p(p(x)) = a\left(ax^2 + bx + c\right)^2 + b\left(ax^2 + bx + c\right) + c$

$$= \left[a^3x^4 + 2a^2bx^3 + a\left(2ac + b^2\right)x^2 + 2abcx + ac^2\right] + \left[abx^2 + b^2x + bc\right] + c$$

$$= a^3x^4 + 2a^2bx^3 + \left[a\left(2ac + b^2\right) + ab\right]x^2 + \left[b\left(2ac + b\right)\right]x + ac^2 + bc + c$$

Hence $2p(p(x)) + 3\left[p(x)\right]^2 - 4p(x)$

$$= \left[a^2(2a+3)\right]x^4 + 2ab(2a+3)x^3 + \ldots + \left[2\left(ac^2 + bc + c\right) + 3c^2 - 4c\right]$$

Therefore $a^2(2a+3) = 1 \Rightarrow 2a^3 + 3a^2 - 1 = 0 \Rightarrow (a+1)^2(2a-1) = 0$

So that $a = -1 \text{ or } \frac{1}{2}$

Now looking at the coefficient of x^3, both these values imply $b = 0$.

Looking at the constant term, $a = -1 \Rightarrow c = 0 \text{ or } 2,\ a = \frac{1}{2} \Rightarrow c = 0 \text{ or } \frac{1}{2}$

Now a solution with $c = 0$ cannot work with $b = 0$ since both $p(p(x))$ and $\left[p(x)\right]^2$ would both be a constant times x^4 and we would have a term in x^2 from the $-4p(x)$ that would not disappear.

Therefore we have 2 possible solutions only :-

$$p(x) = -x^2 + 2 \text{ and } p(x) = \frac{x^2}{2} + \frac{1}{2}$$