Engineering Mathematics 1: ODEs

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Solving ODEs: integrating factors

Sometimes we are fortunate and have an ODE which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x) = g(t)$$

This is called an *exact equation*. Integrating both sides with respect to t gives

$$f(x) = \int g(t)\,\mathrm{d}t$$

For example,

$$t^3 \frac{\mathrm{d}x}{\mathrm{d}t} + 3t^2 x = \mathrm{e}^{2t}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(t^3 x\right) = \mathrm{e}^{2t} \qquad \text{(collect terms)}$$

$$\Rightarrow \quad t^3 x = \frac{1}{2}\mathrm{e}^{2t} + C \qquad \text{(integrate w.r.t. } t\text{)}$$

$$\Rightarrow \quad x(t) = \frac{\mathrm{e}^{2t}}{2t^3} + \frac{C}{t^3} \qquad \text{(solve for } x\text{)}$$

Solving ODEs: integrating factors

It is unusual to find an ODE in *exact form*. However, we can often put an ODE into exact form by multiplying by an *integrating factor*.

Take a first order, linear, homogeneous ODE of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x = 0$$

This can be solved by separation of variables:

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} = -p(t)$$

$$\Rightarrow \quad \int \frac{1}{x}\,\mathrm{d}x = -\int p(t)\,\mathrm{d}t$$

$$\Rightarrow \quad \ln(x(t)) = -\int p(t)\,\mathrm{d}t + C$$

$$\Rightarrow \quad x(t) = \mathrm{e}^{-\int p(t)\,\mathrm{d}t + C}$$

$$\Rightarrow \quad x(t) = A\mathrm{e}^{-\int p(t)\,\mathrm{d}t}$$

Solving ODEs: integrating factors

Thus an equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x = 0$$

can be made exact by multiplying through by $e^{\int p(t)\,\mathrm{d}t}$ to give

$$\begin{aligned} e^{\int p(t)\,\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} + e^{\int p(t)\,\mathrm{d}t}p(t)x &= 0 \\ \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(xe^{\int p(t)\,\mathrm{d}t}\right) &= 0 \\ \Rightarrow \quad xe^{\int p(t)\,\mathrm{d}t} &= A \\ \Rightarrow \quad x &= Ae^{-\int p(t)\,\mathrm{d}t} \end{aligned}$$

For homogeneous equations this technique is of limited use, but for non-homogeneous equations of the (inseparable) form

$$\frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x = r(t)$$

the *integrating factor* $e^{\int p(t)\,\mathrm{d}t}$ is extremely useful.

Solving ODEs: integrating factors

Example

Take the first order, linear, homogeneous ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{t}x = 0$$

Comparing this against

$$\frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x = 0$$

we see that $p(t) = \frac{1}{t}$, which implies

$$\begin{aligned} x(t) &= Ae^{-\int \frac{1}{t}\,\mathrm{d}t} \\ \Rightarrow \quad &= Ae^{-\ln(t)} \\ \Rightarrow \quad &= \frac{A}{t} \end{aligned}$$

Solving ODEs: integrating factors

Now consider the non-homogeneous case

$$\frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x = r(t)$$

This is slightly more difficult but more useful (can't just use separation of variables).

Remember that by multiplying by the integrating factor we can put the left-hand-side into exact form:

$$\begin{aligned} e^{\int p(t)\,\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} + e^{\int p(t)\,\mathrm{d}t}p(t)x &= e^{\int p(t)\,\mathrm{d}t}r(t) \\ \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(xe^{\int p(t)\,\mathrm{d}t}\right) &= e^{\int p(t)\,\mathrm{d}t}r(t) \\ \Rightarrow \quad xe^{\int p(t)\,\mathrm{d}t} &= \int e^{\int p(t)\,\mathrm{d}t}r(t)\,\mathrm{d}t + C \\ \Rightarrow \quad x(t) &= e^{-\int p(t)\,\mathrm{d}t}\left(\int e^{\int p(t)\,\mathrm{d}t}r(t)\,\mathrm{d}t + C\right) \end{aligned}$$

Solving ODEs: integrating factors

Example

Consider the linear, non-homogeneous, first-order ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{t}x = t^2 \qquad \text{with} \qquad x(2) = \frac{1}{3}$$

Clearly $p(t) = \frac{1}{t}$ and so the integrating factor is

$$\text{IF} = \mathrm{e}^{\int p(t)\,\mathrm{d}t} = \mathrm{e}^{\int \frac{1}{t}\,\mathrm{d}t} = \mathrm{e}^{\ln(t)+C} = A\mathrm{e}^{\ln(t)} = At$$

Multiply through by the integrating factor

$$At\frac{\mathrm{d}x}{\mathrm{d}t} + At\frac{1}{t}x = At^3$$

Notice that the integration constant factors out; *this is always the case*, thus we can ignore the integration constant when calculating the integrating factor.

Solving ODEs: integrating factors

This is now an *exact equation*

$$t\frac{\mathrm{d}x}{\mathrm{d}t} + x = \frac{\mathrm{d}}{\mathrm{d}t}(tx) = t^3$$

Thus the general solution is

$$tx = \int t^3\,\mathrm{d}t = \frac{1}{4}t^4 + C$$
$$\Rightarrow \qquad x(t) = \frac{1}{4}t^3 + \frac{C}{t}$$

Finally, use the initial condition to calculate the value of C

$$x(2) = \frac{1}{4}2^3 + \frac{C}{2} = \frac{1}{3}$$
$$\Rightarrow \qquad C = -\frac{10}{3}$$

Thus the particular solution is

$$x(t) = \frac{1}{4}t^3 - \frac{10}{3t}$$

Solving ODEs: integrating factors

Example

Consider the linear, first-order, non-homogeneous ODE

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy = x + 1 \qquad \Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \frac{1}{x} + \frac{1}{x^2}$$

Thus the integrating factor is

$$\text{IF} = \mathrm{e}^{\int \frac{1}{x}\,\mathrm{d}x} = \mathrm{e}^{\ln(x)} = x$$

Thus

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{\mathrm{d}}{\mathrm{d}x}(xy) = 1 + \frac{1}{x}$$
$$\Rightarrow \qquad xy = \int 1 + \frac{1}{x}\,\mathrm{d}x = x + \ln(x) + C$$
$$\Rightarrow \qquad y(x) = 1 + \frac{\ln(x) + C}{x}$$

Solving ODEs: integrating factors

Exercise

Classify and solve

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + tx = t$$

Solving ODEs: integrating factors

James, *Modern Engineering Mathematics*

Read sections 10.5.9.

Attempt a selection of exercises from 10.5.11, questions 31 to 34.

Solving ODEs: other exact equations

Integrating factors give us a straightforward way to solve linear, first-order differential equations. However, exact equations may also be non-linear and not amenable to integrating factors.

Thus, if we can *spot* that a non-linear equation is exact, we have an easy way of solving it. Consider the general first-order ODE

$$q(x,t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + p(x,t) = 0$$

If the equation is exact, then there exists a function $h(x,t)$ such that

$$\frac{\mathrm{d}\,h}{\mathrm{d}\,t} = 0 \qquad \text{such that} \qquad \frac{\partial h}{\partial x} = q(x,t) \quad \text{and} \quad \frac{\partial h}{\partial t} = p(x,t)$$

(Remember that

$$\frac{\mathrm{d}\,h}{\mathrm{d}\,t} = \frac{\partial h}{\partial x}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \frac{\partial h}{\partial t}$$

is the link between *total derivatives* and *partial derivatives*.)

Solving ODEs: other exact equations

Clearly, if we can find such a function $h(x, t)$ then the solution to this (potentially non-linear) ODE is easy

$$\frac{\mathrm{d}\,h}{\mathrm{d}\,t} = 0 \qquad \Rightarrow \qquad h(x,t) = C$$

It is then just a case of rearranging to find x.

Solving ODEs: other exact equations

A simple example of this is

$$t^2 \frac{\mathrm{d}\,x}{\mathrm{d}\,t} + 2xt = \cos(t)$$

It is possible to solve this equation using an integrating factor (since it is linear) to find that

$$h(x,t) = xt^2 - \sin(t)$$

Check:

$$\frac{\partial h}{\partial x} = q(x,t) = t^2 \qquad \text{and} \qquad \frac{\partial h}{\partial t} = p(x,t) = 2xt - \cos(t)$$

Solving ODEs: other exact equations

For a given ODE, how do we determine if it is in exact form?

Check the partial derivatives; if $h(x, t)$ is continuous then we know that

$$\frac{\partial^2 h}{\partial x \partial t} = \frac{\partial^2 h}{\partial t \partial x}$$

Since we have $\frac{\partial h}{\partial x} = q(x,t)$ and $\frac{\partial h}{\partial t} = p(x,t)$ we can calculate both second derivatives and check that they are equal; i.e.,

$$\frac{\partial q}{\partial t} = \frac{\partial p}{\partial x}$$

where

$$q(x,t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + p(x,t) = 0$$

If the two derivatives are not equal, then the equation is not exact.

Solving ODEs: other exact equations

Example

Consider the non-linear, non-homogeneous, first-order ODE

$$(x+t)\frac{\mathrm{d}x}{\mathrm{d}t} - x = -t$$

In this case

$$q(x,t) = x+t \qquad \text{and} \qquad p(x,t) = t-x$$

Check that this is an exact equation

$$\frac{\partial q}{\partial t} = 1 \qquad \text{and} \qquad \frac{\partial p}{\partial x} = -1$$

It isn't so stop.

Solving ODEs: other exact equations

Example

Consider the non-linear, non-homogeneous, first-order ODE

$$\frac{t+1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} + \ln(x) = \cos(t)$$

In this case

$$q(x,t) = \frac{t+1}{x} \qquad \text{and} \qquad p(x,t) = \ln(x) - \cos(t)$$

Check that this is an exact equation

$$\frac{\partial q}{\partial t} = \frac{1}{x} \qquad \text{and} \qquad \frac{\partial p}{\partial x} = \frac{1}{x}$$

It is, so now we need to calculate $h(x,t)$.

Solving ODEs: other exact equations

Now we integrate $\frac{\partial h}{\partial x}$ with respect to x

$$h(x,t) = \int q(x,t)\,\mathrm{d}x = \int \frac{t+1}{x}\,\mathrm{d}x = (t+1)\int \frac{1}{x}\,\mathrm{d}x$$
$$= (t+1)\,[\ln(x) + C(t)]$$

The key thing to remember is that C is a function of t and we need to determine it.

Now we integrate $\frac{\partial h}{\partial t}$ with respect to t

$$h(x,t) = \int p(x,t)\,\mathrm{d}t = \int \ln(x) - \cos(t)\,\mathrm{d}t = \ln(x)\,t - \sin(t) + D(x)$$

This time, since we integrated with respect to t, the integration constant is a function of x. Now we compare the two expressions

$$\ln(x)\,t - \sin(t) + D(x) = (t+1)\,[\ln(x) + C(t)]$$

Solving ODEs: other exact equations

$$\ln(x)\,t - \sin(t) + D(x) = (t+1)\,[\ln(x) + C(t)]$$

Since $D(x)$ cannot contain terms involving t and, similarly, $C(t)$ cannot contain terms involving x, we can determine (up to a constant) C and D uniquely.

In this case

$$D(x) = \ln(x) \qquad \text{and} \qquad C(t) = -\frac{\sin(t)}{t+1}$$

to give an *implicit solution* of

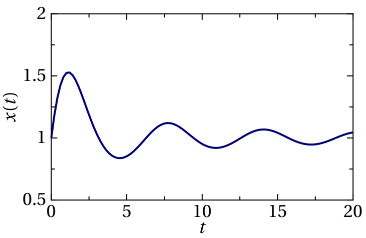
$$h(x,t) = (t+1)\ln(x) - \sin(t) = E$$

Thus, the *general solution* is

$$x(t) = \mathrm{e}^{\frac{E+\sin(t)}{t+1}}$$

Solving ODEs: other exact equations

$$x(t) = \mathrm{e}^{\frac{\sin(t)}{t+1}} \qquad (E = 0 \text{ when } x(0) = 1)$$



Solving ODEs: other exact equations

Exercise

Determine if the following ODE is exact and, if so, find its solution

$$\cos(t)\frac{\mathrm{d}x}{\mathrm{d}t} - x\sin(t) = 1$$

Solving ODEs: other exact equations

James, *Modern Engineering Mathematics*

Read sections 10.5.7.

Attempt a selection of exercises from 10.5.8.