MEI Core 4

Further techniques for integration

Chapter assessment

1. Find the volume of the solid generated when the area bounded by $y = 3x - x^2$, the x axis and the lines $x = 1$ and $x = 2$ is rotated through 360° about the x axis. [5]

2. Find the volume formed by rotating completely about the x axis the area bounded by $y = x^3 - 2x^2$ and the x axis. [5]

3. The area formed by $y = \sqrt{x}$, $y = 2$ and the y axis is rotated through 360° about the y axis. Find the volume generated. [4]

4. Find $\int_0^{\pi/2} \frac{\cos x}{\sin x + 1}\,dx$. [3]

5. Express $f(x) = \frac{x}{(x+1)(x+2)}$ in partial fractions and hence evaluate $\int_0^2 f(x)\,dx$ leaving your answer in logarithmic form. [6]

6. Using a suitable method integrate
   (i) $\int \frac{x}{(x^2-1)^3}\,dx$. [4]
   (ii) $\int \frac{x}{x-1}\,dx$ [4]
   (iii) $\int_0^2 xe^{x^2}\,dx$ [4]

7. (i) Use the trapezium rule to obtain an approximate value for $\int_1^3 \frac{1}{1+x}\,dx$ with
   (a) 4 strips b) 8 strips. [10]
   (ii) Use integration to obtain an exact value for the integral. [4]
   (iii) Calculate the percentage errors for each of the answers calculated using the trapezium rule. [2]

8. (i) Show that $\frac{x^2+1}{x(x-1)} = 1 + \frac{x+1}{x(x-1)}$. [2]
   (ii) Write $\frac{x+1}{x(x-1)}$ as partial fractions.
   Hence show that $\int_2^3 \frac{x^2+1}{x(x-1)}\,dx = 1 + \ln\frac{8}{3}$. [7]

Total 60 marks

MEI C4 Integration Assessment solutions

Further techniques for integration

Solutions to Chapter assessment

1. Volume $= \int_1^2 \pi y^2 dx$

$$= \pi \int_1^2 (3x - x^2)^2 dx$$

$$= \pi \int_1^2 (9x^2 - 6x^3 + x^4) dx$$

$$= \pi \left[3x^3 - \tfrac{3}{2}x^4 + \tfrac{1}{5}x^5 \right]_1^2$$

$$= \pi \left(24 - 24 + \tfrac{32}{5} - (3 - \tfrac{3}{2} + \tfrac{1}{5}) \right)$$

$$= \tfrac{47}{10}\pi$$

[5]

2. $y = x^3 - 2x^2$

When $y = 0$, $x^2(x-2) = 0$

$x = 0$ or $x = 2$

Volume $= \int_0^2 \pi y^2 dx$

$$= \pi \int_0^2 (x^3 - 2x^2)^2 dx$$

$$= \pi \int_0^2 (x^6 - 4x^5 + 4x^4) dx$$

$$= \pi \left[\tfrac{1}{7}x^7 - \tfrac{2}{3}x^6 + \tfrac{4}{5}x^5 \right]_0^2$$

$$= \pi \left(\tfrac{128}{7} - \tfrac{128}{3} + \tfrac{128}{5} \right)$$

$$= \tfrac{128}{105}\pi$$

[5]

3. $y = \sqrt{x} \Rightarrow x = y^2$

Volume $= \int_0^2 \pi x^2 dy$

$$= \pi \int_0^2 (y^2)^2 dy$$

$$= \pi \int_0^2 y^4 \, dy$$

$$= \pi \left[\tfrac{1}{5}y^5 \right]_0^2$$

$$= \pi \left(\tfrac{32}{5} - 0 \right)$$

$$= \tfrac{32}{5}\pi$$

[4]

4. The derivative of $\sin x + 1$ is $\cos x$, so this integral can be done by inspection. (Alternatively, use the substitution $u = \sin x + 1$).

$$\int_0^{\pi/2} \frac{\cos x}{\sin x + 1}\,dx = \left[\ln(\sin x + 1)\right]_0^{\pi/2}$$
$$= \ln 2 - \ln 1$$
$$= \ln 2$$

[3]

5. $$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$
$$x = A(x+2) + B(x+1)$$
Putting $x = -2 \quad \Rightarrow -2 = -B \quad \Rightarrow B = 2$

Putting $x = -1 \quad \Rightarrow -1 = A \quad \Rightarrow A = -1$

$$\frac{x}{(x+1)(x+2)} = \frac{2}{x+2} - \frac{1}{x+1}$$
$$\int_0^2 f(x)\,dx = \int_0^2 \left(\frac{2}{x+2} - \frac{1}{x+1}\right)dx$$
$$= \left[2\ln(x+2) - \ln(x+1)\right]_0^2$$
$$= 2\ln 4 - \ln 3 - (2\ln 2 - \ln 1)$$
$$= \ln \frac{4^2}{3 \times 2^2}$$
$$= \ln \tfrac{4}{3}$$

[6]

6. (i) The derivative of $x^2 - 1$ is $2x$, so this integral can be done by inspection. (Alternatively, use the substitution $u = x^2 - 1$).

$$\int \frac{x}{(x^2-1)^3}\,dx = \tfrac{1}{2}\int 2x(x^2-1)^{-3}\,dx$$
$$= \tfrac{1}{2} \times -\tfrac{1}{2}(x^2-1)^{-2} + c$$
$$= -\frac{1}{4(x^2-1)^2} + c$$

[4]

(ii) Let $u = x - 1 \quad \Rightarrow \frac{du}{dx} = 1$

$$\int \frac{x}{x-1}\,dx = \int \frac{u+1}{u}\,du$$
$$= \int \left(1 + \frac{1}{u}\right)du$$
$$= u + \ln u + c$$
$$= x - 1 + \ln(x-1) + c$$
$$= x + \ln(x-1) + k$$

Replacing c – 1 with a new constant k

[4]

(iii) The derivative of x^2 is $2x$, so this integral can be done by inspection. (Alternatively, use the substitution $u = x^2$).

$$\int_0^2 xe^{x^2}dx = \tfrac{1}{2}\int_0^2 2xe^{x^2}dx$$
$$= \tfrac{1}{2}\left[e^{x^2}\right]_0^2$$
$$= \tfrac{1}{2}\left(e^4 - 1\right)$$

[4]

7\. (i) (a) For 4 strips, $h = \dfrac{3-1}{4} = 0.5$

$$\int_1^3 \frac{1}{1+x}\,dx \approx \tfrac{1}{2}h\left[f(1) + 2\left(f(1.5) + f(2) + f(2.5)\right) + f(3)\right]$$
$$= 0.25\left[\tfrac{1}{2} + 2\left(\tfrac{1}{2.5} + \tfrac{1}{3} + \tfrac{1}{3.5}\right) + \tfrac{1}{4}\right]$$
$$= 0.6970$$

(b) For 8 strips, $h = \dfrac{3-1}{8} = 0.25$

$$\int_1^3 \frac{1}{1+x}\,dx \approx \tfrac{1}{2}h\left[f(1) + 2\left(f(1.25) + f(1.5) + \ldots + f(2.75)\right) + f(3)\right]$$
$$= 0.125\left[\tfrac{1}{2} + 2\left(\tfrac{1}{2.25} + \tfrac{1}{2.5} + \tfrac{1}{2.75} + \tfrac{1}{3} + \tfrac{1}{3.25} + \tfrac{1}{3.5} + \tfrac{1}{3.75}\right) + \tfrac{1}{4}\right]$$
$$= 0.6941$$

[10]

(ii) $$\int_1^3 \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_1^3$$
$$= \ln 4 - \ln 2$$
$$= \ln \tfrac{4}{2}$$
$$= \ln 2$$

[4]

(iii) (a) Percentage error $= \dfrac{0.6971 - \ln 2}{\ln 2} \times 100 = 0.57\%$

(a) Percentage error $= \dfrac{0.6941 - \ln 2}{\ln 2} \times 100 = 0.14\%$

[2]

8\. (i) $$\frac{x^2+1}{x(x-1)} = \frac{x(x-1) + x + 1}{x(x-1)}$$
$$= \frac{x(x-1)}{x(x-1)} + \frac{x+1}{x(x-1)}$$
$$= 1 + \frac{x+1}{x(x-1)}$$

[2]

(ii) $$\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$
$x + 1 = A(x-1) + Bx$

Putting $x = 0 \quad \Rightarrow 1 = -A \quad \Rightarrow A = -1$

Putting $x=1 \quad \Rightarrow 2=B$

$$\frac{x+1}{x(x-1)} = \frac{2}{x-1} - \frac{1}{x}$$

$$\begin{aligned}
\int_2^3 \frac{x^2+1}{x(x-1)}\,dx &= \int_2^3 \left(1+\frac{2}{x-1}-\frac{1}{x}\right)dx \\
&= \left[x+2\ln(x-1)-\ln x\right]_2^3 \\
&= 3+2\ln 2-\ln 3-(2+2\ln 1-\ln 2) \\
&= 3+2\ln 2-\ln 3-2-0+\ln 2 \\
&= 1+3\ln 2-\ln 3 \\
&= 1+\ln\frac{8}{3}
\end{aligned}$$

[7]