OXFORD ENTRANCE PRACTICE

SHEET 1

1. Prove that the sum of the first n terms of the artithmetical progression

$$a + (a + d) + (a + 2d) + \ldots,$$

is

$$\tfrac{1}{2}n\{2a + (n - 1)d\}.$$

Hence, or otherwise, show that

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2n - 1)^2 - (2n)^2 = -n(2n + 1).$$

Deduce the sums of the series

(*a*) $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2n - 1)^2 - (2n)^2 + (2n + 1)^2$,

and

(*b*) $21^2 - 22^2 + 23^2 - 24^2 + \ldots + 39^2 - 40^2$.

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2. The point O is the origin of coordinates, A is the point $(-5, 5)$ and C is the point $(7, 1)$. Find, by calculation, the coordinates of the point B such that both $AB = BC$ and also angle OAB is a right angle.

Prove that O, A, B, C lie on a circle and find its equation.

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3. (i) Prove that, if x and $\sqrt{(1 - x)}$ are real,

$$x + \sqrt{(1 - x)} \leqslant \tfrac{5}{4}.$$

(ii) Find the least value of

$$f(x) \equiv \sqrt{(1 - x)} + \sqrt{(4 - x)},$$

given that x and $f(x)$ are real.

(iii) If x and y are real, sketch the whole of the curve

$$y = \sqrt{(1 - x^2)} + \sqrt{(4 - x^2)}.$$

[The symbol $\sqrt{(u)}$ denotes the non-negative value of the square root of u.]

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4. Sketch the curve

$$y = \frac{(x - 1)(x - 4)}{(x - 2)(x - 3)}$$

and also, using this cuve if you wish and explaining your argument carefully, the curve

$$y^2 = \frac{(x - 1)(x - 4)}{(x - 2)(x - 3)}.$$

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5. Indicate in a diagram the positions of the points A_0, A_1, A_2, A_3 whose coordinates referred to rectangular Cartesian axes Ox, Oy are

$$A_0(1, 1),\quad A_1(\tfrac{1}{2}, \tfrac{1}{3}),\quad A_2(\tfrac{1}{4}, \tfrac{1}{9}),\quad A_3(\tfrac{1}{8}, \tfrac{1}{27}).$$

Consider the $(n + 1)$ points $A_p(2^{-p}, 3^{-p})$, where

$$p = 0, 1, 2, \ldots, n.$$

(i) Find the limit as $n \to \infty$ of the sum of the areas of the n rectangles having sides parallel to Ox, Oy and opposite vertices at adjacent points A_p, A_{p+1} [The area of the first such rectangle, with $p = 0$, is $\tfrac{1}{3}$.]

(ii) Find the limit as $n \to \infty$ of the sum of the areas of the circles circumscribing these rectangles. [The area of the first such circle is $\tfrac{25}{144}\pi$.]

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6. Draw a diagram to illustrate, with respect to rectangular Cartesian axes, the square whose sides are $x = 4\pi$, $x = -4\pi$, $y = 4\pi$, $y = -4\pi$.

Indicate all the points inside the square for which

$$\sin y = \sin x,$$

showing that they consist of two isolated points (at two of the vertices) and a number of straight lines of total length $64\pi\sqrt{2}$. [The argument giving this number must be clearly stated.]