OXFORD ENTRANCE PRACTICE SHEET 2

(1) Given that $2y = a^x + a^{-x}$, where $a > 1$, $x > 0$, prove that
$$a^x = y + \sqrt{(y^2 - 1)}.$$
If, further, $2z = a^{3x} + a^{-3x}$, prove that
$$z = 4y^3 - 3y.$$

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(2) Draw a sketch-graph of the curve whose equation is
$$y = x^2(2 - x).$$
Hence, or otherwise, draw a sketch-graph of the curve whose equation is
$$y^2 = x^2(2 - x),$$
indicating *briefly* how the form of the curve has been derived.

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(3) Find values of x and y which are positive integers and which satisfy simultaneously the inequalities
$$2x + 3y > 12, \quad x^2 + y^2 < 6x + 4y.$$

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(4) Prove that the equation
$$x^2 + y^2 - 7x - qy + 6 = 0$$
represents the circle through the points (1, 0), (6, 0) having its centre at the point $A\ (\tfrac{7}{2}, \tfrac{1}{2}q)$; and find the equation of the circle through the points (0, 2), (0, 3) having its centre at the point $B\ (\tfrac{1}{2}p, \tfrac{5}{2})$.

Assuming that the circles intersect, prove that
$$(p - 7)x = (q - 5)y$$
is the equation of the chord common to the two circles. Discuss briefly the case $p = 7$, $q = 5$.

Prove also that, if the circles are so related that the sum of the squares of their radii is equal to AB^2, then $7p + 5q = 24$.

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(5) Sketch the curve
$$y = x(x - 1)(x - \lambda)$$
for each of the cases
$$\lambda < 0, \quad \lambda = 0, \quad 0 < \lambda < 1, \quad \lambda = 1, \quad \lambda > 1.$$
[Only the general shape is required. In particular, you need not work out exact positions or values for the maxima and minima of y.]

The turning points of the curve are at $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$. Prove that
$$x_1 + x_2 = \tfrac{2}{3}(\lambda + 1), \quad x_1 x_2 = \tfrac{1}{3}\lambda,$$
and deduce that the gradient of the line AB is negative for all values of λ.

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(6) The points A, B have coordinates $(a, 0)$ and $(-a, 0)$ respectively. A point P in the plane of the coordinate axes moves so that $AP : PB = \lambda$, where λ is a positive constant. Find the equation of the locus of P, and show that, if $\lambda \neq 1$, this locus is a circle. Show that the centre or this circle lies on the axis of x, but not between A and B.

Find the value of λ if the centre of the circle is at the point C on the axis of x, where $CB : CA = 3 : 5$.

$\frac{AP}{PB} = \frac{\lambda}{1}$

$AP^2 = \lambda^2 PB^2$

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(7) Find all integer values of x and y that satisfy simultaneously the inequalities
$$x + 2y \geqslant 2,$$
$$-x + 3y \leqslant 3,$$
$$3x - 4y \leqslant 6.$$

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(8) Solve completely the simultaneous equations
$$x + y + z = 3,$$
$$x + 2y + 4z = 7,$$
$$x + ky + k^2 z = 1 + k + k^3.$$
Your solution should give (i) expressions for x, y, z for a general value of k; (ii) formulae giving all values of x, y, z for those values of k for which the solution is not unique; (iii) any value of k for which the equations have no solution.

(Be careful to copy the third equation correctly.)