OXFORD ENTRANCE PRACTICE SHEET 3

(1) The sum of the first n terms of an arithmetic progression is denoted by the symbol S_n. Prove that

$$(q-r)\frac{S_p}{p}+(r-p)\frac{S_q}{q}+(p-q)\frac{S_r}{r}=0.$$

Deduce, or prove otherwise, that if, $p+q=2r$, then

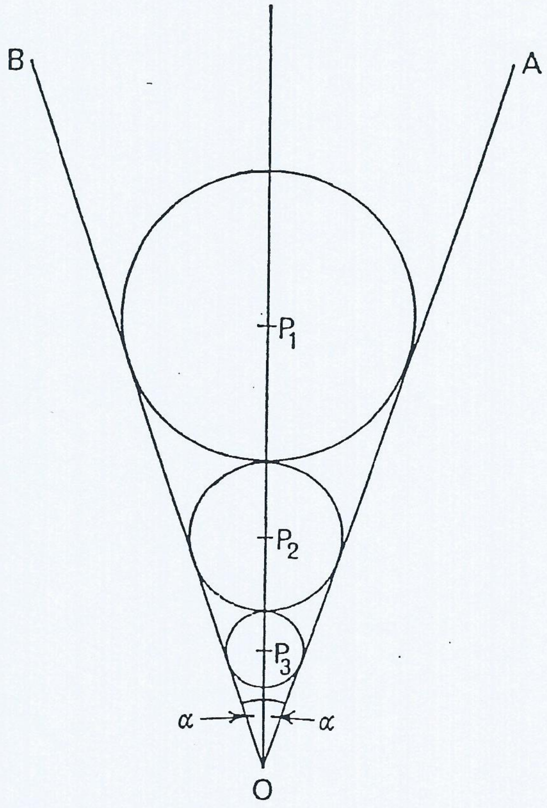
$$\frac{S_p}{p}+\frac{S_q}{q}=2\frac{S_r}{r}.$$

(2) Find the coordinates of the points common to the curves

$$y=x^2-1,$$
$$y=(x^2-1)^3.$$

Sketch these curves in the same diagram.

(3)



The diagram represents two straight lines OA, OB inclined at an angle 2α. The circle of centre P_1 has radius a and touches each of OA, OB. A sequence of circles is drawn, decreasing in radius, each touching OA, OB and its immediate predecessor. Prove that the areas of these circles are in geometric progression.

The sum of the first n of these areas is S_n and the sum to infinity of the geometric progression is S. Prove that the difference $S-S_n$ is less than $\frac{1}{100}S$ whenever n exceeds

$$1/\lg\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right). \quad [\lg x \equiv \log_{10}x.]$$

Prove also that the area of the first circle is equal to the sum of the areas of all the other circles when $\sin\alpha = 3-2\sqrt{2}$.

(4) (*a*) Indicate in a diagram the points in the (x, y)-plane whose coordinates lie in the ranges

$$-2\pi \leqslant x \leqslant 2\pi, \quad -2\pi \leqslant y \leqslant 2\pi$$

and satisfy the equation

$$\sin^2 x+\cos^2 y=2.$$

(*b*) Find all the (real) values of x and y that satisfy the simultaneous equations

$$16\sin x+9\tan y=35,$$
$$64\sin^2 x+27\tan^2 y=259,$$

where x, y are the measures of the angles expressed in degrees.