OXFORD ENTRANCE PRACTICE SHEET 4

1 (*a*) Prove that
$$\log_y x = \frac{1}{\log_x y},$$
where $x > 0$ and $y > 0$.
Find the possible values of x if
$$2(\log_9 x + \log_x 9) = 5.$$
(*b*) Solve the equation
$$\sqrt{(3x-5)} - \sqrt{(x+2)} = 1,$$
where the positive values of the square roots are to be taken.

2 (*a*) In a given plane there are four given points on a straight line l and four other points, no three of which are collinear, and no two of which are collinear with any of the given points on l. Show that, in addition to l, 22 lines can be obtained by joining up the points in pairs.

Find the number of triangles that can be formed with vertices at three of the points. Make your method of calculation clear.
(*b*) Show, by putting $a = b+c$ or otherwise, that $a-b-c$ is a factor of
$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2.$$
Factorise this expression completely.

3 (*a*) If the first, fifth and tenth terms of an arithmetic progression are in geometric progression and the sum of the second and eighth terms is 20, find the first term and the (non-zero) common difference.
(*b*) Find how many *even* numbers between 3000 and 7000 can be written down using the digits 1, 3, 6, 8

(i) if no digit can occur more than once in any number,
(ii) if repetition of digits is allowed.

4 (*a*) Solve, for values of x between 0° and 360° inclusive, the equations
(i) $\cos 2x = \cos x$,
(ii) $\sin x = 2\sin(60° - x)$.
(*b*) If $p\cos 2x + q\sin 2x + r = 0$, where $p \neq 0$ and $p \neq r$, find an equation for $\tan x$.
Deduce that, if the roots of this equation are $\tan x_1$ and $\tan x_2$, then
$$\tan(x_1+x_2) = \frac{q}{p}.$$

5 The vertices of triangle ABC are $A(-16, 0)$, $B(9, 0)$ and $C(0, 12)$.
✓ Prove that the equation of the internal bisector of the angle A of the triangle is $x-3y+16 = 0$.
Find the equation of the internal bisector of the angle B of the triangle.
Hence, or otherwise, find the equation of a circle which touches all three sides of the triangle.

6 Prove that
$$a^2+b^2+c^2-bc-ca-ab = \tfrac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$
Given that
$$bc+ca+ab = k(a^2+b^2+c^2),$$
prove that $k \leqslant 1$. Under what conditions is $k = 1$?
If, further, a, b, c are the lengths of the sides of a triangle, prove that $k > \frac{1}{2}$.
By considering triangles with $b = c = 1$, or otherwise, prove that it is possible to obtain triangles in which k is arbitrarily close to $\frac{1}{2}$.