Notes For Core Mathematics C1

ALGEBRA

Vocabulary:

The list is not exhaustive but covers most of the terms you will meet in the early part of your course.

Expression- Any collection of numbers, letters and symbols.

Constant- A value that does not change.

Variable- A letter that can represent various values.

Term- Any collection of numbers and letters. e.g. $3x^2yz,\ 2\sqrt{pqr}\ ,\ \frac{7abc}{d^2}$

Coefficient- The numerical part of a term, e.g. In the above terms 3 is the coefficient of x^2yz.

Function- A rule which enables us to find the value of one variable from those of others together with a specification of the values those functions can take. (The domain of the function)

e.g. $f(x) = 3x^2 - 4x + 7$ for $x > 5$

Note that the statement of the domain is often omitted if it is obvious what is intended.

Index- The power to which a number or variable is to be raised. e.g. $3^5 = 3\times3\times3\times3\times3,\ x^3 = x\times x\times x$

Note that it tells us how many of the items are to be multiplied together. Not the number of multiplications.

Linear- An expression in which the variables have no powers higher that 1. e.g $2x - 3y + 7$ but not $3x^2 + 2$

Identity- A statement that is true for ALL values of the variables involved e.g. $2(x+y) = 2x + 2y$

Equation- A statement that is true for only a limited number of values of the variables. e.g. $4x = 8$

Polynomial An expression of the form $a_0 + a_1x + a_2x^2 + a_3x^3 .... + a_nx^n$ where the $a's$ are constants,with $a_n \neq 0$ is called a POLYNOMIAL in x of degree n

Solution of Equations

(1) Linear equations in one unknown. This simply involves rearranging the equation until you have the unknown quantity on its own on one side. This rearrangement is achieved by

(i) adding/subtracting the same quantity to/from both sides of the equation.

(ii) multiplying or dividing both sides by the same quantity. But be very careful if multiplying or dividing by a negative number and be very wary of dividing by anything that could possibly have a zero value.

Remember! Whatever you do to one side of an equation, you must also do to the other side.

Ex. Solve $2(x+3) = 7 - 2x$

(1) We first remove brackets to give $2x + 6 = 7 - 2x$

It is usual practice to rearrange with the unknown on the left hand side so

(2) We remove the 6 from the left hand side by subtracting 6 from both sides $2x = 1 - 2x$

(3) We then remove the $2x$ from the right hand side by adding $2x$ to both sides $4x = 1$

(4) Finally we divide both sides by 4 to give $x = \frac{1}{4}$

Steps (2) and (3) would usually be done at the same time.

Quadratic equations

These are equations in which we have the square of the unknown quantity. There are three common methods of solution. Before we look at the first method however we must consider the technique of factorising an expression

Factorising Polynomials

Two polynomials may be added or subtracted by adding or subtracting corresponding terms.

Ex. $(a_0 + a_1x + a_2x^2 + a_3x^3...) \pm (b_0 + b_1x + b_2x^2 + b_3x^3...) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + ...$

To multiply two polynomials, (or any two algebraic expression) each term in one must be multiplied by each term in the other.

Ex. $(ax+by)(cx+dy) = ax(cx+dy) + by(cx+dy) = acx^2 + adxy + bcxy + bdy^2 = acx^2 + (ad+bc)xy + bdy^2$

Notice how like terms, such as the two xy terms, should be collected together, especially if they have numerical coefficients.

Ex. $(3x+2y)(2x-5y) = 3x(2x-5y) + 2y(2x-5y) = 6x^2 - 15xy + 4xy - 10y^2 = 6x^2 - 11xy - 10y^2$

Notice that we must obey the usual rules with regard to signs. Make sure you can recognise the patterns in the results as this will help you later. When multiplying two '2 term expressions' as above, the result is given by 'product of first terms plus inner and outer products combined plus product of last terms'

Ex. Consider $(7x-3y)(2x+5y)$ The product of the first terms is $7x \times 2x = 14x^2$ The inner product is $(-3y) \times 2x = -6xy$ and the outer product is $7x \times 5y = 35xy$ so combined we have $29xy$ and the product of the last terms is $(-3y) \times 5y = -15y^2$ Thus $(7x-3y)(2x+5y) = 14x^2 + 29xy - 15y^2$
A special case is when we have to square a two term expression. It should be fairly obvious that the inner and outer products are then going to be the same and so the result is given by 'square of first term plus twice the product of the terms plus square of the last'

Ex. $(3x+4y)^2 = (3x)^2 + 2 \times 3x \times 4y + (4y)^2 = 9x^2 + 24xy + 16y^2$
$(5x-2)^2 = (5x)^2 + 2 \times 5x \times (-2) + (-2)^2 = 25x^2 - 20x + 4$
Another very special case is when we have the same two terms in each bracket but with opposite signs.
Ex. $(x+3)(x-3) = x^2 + 3x - 3x - 3^2 = x^2 - 9$ and we see that the inner and outer products cancel each other resulting in the 'difference of two squares'

The reverse process to multiplying two polynomials together is called FACTORISING. There are a number of simple rules that you must learn and apply.

(1) ALWAYS look for a COMMON FACTOR first. ie.. a number or letter or combination of numbers and letters that will divide exactly into every term of the polynomial.
Ex. $6a+8b$ has a common factor of 2 so $6a+8b = 2(3a+4b)$
Ex. $3x^3 - 6x^2 + 15x = 3x\left(x^2 - 2x + 5\right)$
Failure to do this will make it more difficult to recognise subsequent factors.

(2) The only 2-term expression that you can factorise (other than a common factor) is the difference of two squares.
Ex. $4x^2 - 9$ This is clearly the difference of two squares and your earlier experience of multiplying out should help you to realise that the factors are $2x+3$ and $2x-3$ i.e. the sum of the square roots of the two terms and the difference of the square roots. Whenever you see the difference of two perfect squares you should immediately be able to write it in factored form.
Ex. $121x^4y^2 - 36z^6 = \left(11x^2y + 6z\right)\left(11x^2y - 6z\right)$
Ex. $27x^3 - 3xy^2$ This is an example where , if you fail to spot the common factor of $3x$ it may prevent you from recognising the difference of two squares, ie $27x^3 - 3xy^2 = 3x\left(9x^2 - y^2\right) = 3x(3x+y)(3x-y)$
Recognising this type of expression can also be a great help in performing certain calculations
Ex. $101^2 - 99^2 = (101+99)(101-99) = 200 \times 2 = 400$

(3) 3-term expressions. From our earlier work on multiplying we know that a 3-term expression might factorise into the product of two 2-term expressions, in which case we can reverse some of the 'rules' established then.
Ex. Factorise $3x^2 - xy - 2y^2$ The earlier patterns we obtained tell us that we are looking for a result of the form $(ax+by)(cx+dy)$ where we must have $ac = 3,\ bd = -2$ and $bc + ad = -1$ We are fortunate here in that there are not many possibilities to consider. Clearly a and c must be 3 and 1 (or 1 and 3) and similarly, b and d must be 2 and 1 or 1 and 2 with one of them positive and the other negative. At the worst then we have to consider $(3x+2y)(x-y),\ (3x-2y)(x+y),\ (3x+y)(x-2y)$, and $(3x-y)(x+2y)$,
The fact that the inner and outer products must combine to give $-xy$ enable us to see that it is the first of these possibilities that we require. So $3x^2 - xy - 2y^2 = (3x+2y)(x-y)$
Now consider a much more difficult example: $6x^2 + 5x - 6$ First note that there are no common factors. Next, the negative last term tells us that we want different signs in the two brackets. A consideration of how the inner and outer products combine then tells us that we want these two products to have a numerical difference of $5x$ with the larger one being positive. Now we begin to consider the various possibilities. The first terms can apparently be $6x$ and x or $3x$ and $2x$ and likewise the last terms can be 6 and 1 or 2 and 3. However since there were no common factors then we cannot have a common factor in either of our brackets which means that the only possibilities are $(6x...1)(x...6), (3x...2)(2x...3)$
and it is easy then to see that it must be $(3x-2)(2x+3)$
Ex. $8x^2 - 6x - 9$ A useful way of setting down the trials is with the two brackets forming the rows of a matrix thus $\begin{pmatrix} 4x & 3 \\ 2x & 3 \end{pmatrix}, \begin{pmatrix} 8x & 3 \\ x & 3 \end{pmatrix}, \begin{pmatrix} 4x & 9 \\ 2x & 1 \end{pmatrix}, \begin{pmatrix} 2x & 9 \\ 4x & 1 \end{pmatrix}, \begin{pmatrix} 8x & 9 \\ x & 1 \end{pmatrix}, \begin{pmatrix} x & 9 \\ 8x & 1 \end{pmatrix}$ The inner and outer products are

then given by the 'cross products' in the matrices so we require the difference (because of the negative last term) of the cross products to be 6. By inspection we see that the first possibility is the one we want. ie. $8x^2-6x-9=(4x+3)(2x-3)$. You do not need to write down all of the possible matrices as above. Start with the one you think most likely and then work systematically until you find the correct one. If none work and you are sure you have tried every possible combination then the expression does not factorise. Keep on the look out for perfect squares. i.e. if the first and last terms are perfect squares (and so necessarily positive terms) then check whether the middle term is twice the product of the square roots of the first and last.

Ex. $4x^2-12x+9$ is a perfect square since $12=2\times\sqrt{4}\times\sqrt{9}$ so $4x^2-12x+9=(2x-3)^2$
but $9x^2+12x-4$ is not because the last term is negative
and $36x^2+30x+25$ is not because $2\times\sqrt{36}\times\sqrt{25}=60$ not 30

We now return to the first method of solving a quadratic equation.

(1) By factorising. If we can express the equation in the form $(ax+b)(cx+d)=0$ then we can make use of a fundamental property of numbers which is that if the product of two numbers is zero, then one of them MUST be zero.

Ex. $x^2-2x-3=0 \Rightarrow (x-3)(x+1)=0$ so either $x-3=0$ or $x+1=0 \Rightarrow x=3$ or -1

Ex. $6x^2+5x+3=9 \Rightarrow 6x^2+5x-6=0$ Note that we MUST have zero on the right hand side.
hence, $(3x-2)(2x+3)=0 \Rightarrow x=\frac{2}{3}$ or $-\frac{3}{2}$

(2) By completing the square. In this method we arrange for the unknown to be contained in a perfect square. The examples illustrate the method.

Ex. Solve $x^2-4x-7=0$

Step 1. Isolate the x terms by adding 7 to each side to give $x^2-4x=7$

Step 2. From the pattern of a perfect square we can deduce that x^2-4x can only be obtained by squaring $x-2$ (remember the pattern is 'square of the first plus twice the product of the terms plus square of the last' from which it follows that the second term must be –2) so by adding $(-2)^2$ to both sides we obtain $(x-2)^2=7+4=11$ and we have 'completed the square of the left hand side.

Step 3 Take square roots of both sides $x-2=\pm\sqrt{11} \Rightarrow x=2\pm\sqrt{11}=5.32$ or -1.32

Ex. Solve $5x^2+30x+2=0$ This time we need an addition to the first step because $5x^2$ is not a perfect square.

Step 1 Subtract 2 from each side and divide both sides by 5. $x^2+6x=-\frac{2}{5}$ then proceed as before

$(x+3)^2=-\frac{2}{5}+9=\frac{43}{5} \Rightarrow x+3=\pm\sqrt{\frac{43}{5}} \Rightarrow x=-3\pm\sqrt{8.6}=-0.067$ or -5.93

Note that the technique of completing the square is also extremely useful for locating the line of symmetry or turning point on a quadratic curve or for simply finding the maximum or minimum value of a quadratic expression.

Ex. Find the minimum value of $2x^2-5x-8$

$2x^2-5x-8=2(x^2-\frac{5}{2}x-4)=2[(x-\frac{5}{4})^2-\frac{25}{16}-4]=2(x-\frac{5}{4})^2-\frac{89}{8}$ Since the only part that can vary is in a perfect square then that part has a minimum value of zero (when $x=\frac{5}{4}$) and so the minimum value of the expression is $-\frac{89}{8}$

Ex. Find the coordinates of the maximum turning point on the curve $y=3+6x-x^2$

$3+6x-x^2=-[x^2-6x-3]=-[(x-3)^2-9-3]=-(x-3)^2+12$ Note the first step of making the x^2 term positive. So the expression has a maximum value of 12 when $x=3$ ie the maximum turning point is the point with coordinates (3, 12)

(3) Solving by formula. Applying the method of completing the square to the general quadratic equation $ax^2+bx+c=0$ we may obtain a formula for solving quadratic equations.

$$ax^2+bx+c=0 \Rightarrow x^2+\frac{b}{a}x=-\frac{c}{a} \Rightarrow \left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$$

Hence, $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ You should commit this formula to memory.

Ex. Solve $2x^2-7x-2=0$ We have $a=2,\ b=-7$ and $c=-2$

So applying the formula gives $x=\dfrac{7\pm\sqrt{49+16}}{4}=3.77$ or 0.27

Ex. Solve $3x^2+5x+10=0$ Applying the formula, $x=\dfrac{-5\pm\sqrt{25-120}}{6}=\dfrac{-5\pm\sqrt{-95}}{6}$

Since there is no real number which is the square root of – 95, this equation has no real solutions.
Ex. Solve $2x-x^2=6$ Be particularly careful when the equation does not appear in the standard form.

Rearranging we have $x^2-2x+6=0$ and so $x=\dfrac{2\pm\sqrt{4-24}}{2}$ and there are again no real solutions.

Note that the formula method enables us to see very quickly whether the equation has any solutions or not. All we need to do is check the value inside the square root i.e. b^2-4ac and if this is not a positive number then the equation has no real solutions. In general if

$b^2-4ac>0$ there are two distinct solutions, $b^2-4ac=0$ there is only one solution i.e. a repeated root and if $b^2-4ac<0$ there are no real solutions.

4. Graphical Solution

This is an acceptable method if you only require an approximate solution but cannot be relied upon to give an accurate one.

The method is quite simple, to solve an equation of the form $y=f(x)$ where $f(x)$ is any function of x, we simply plot the graph of $f(x)=0$ and read off where it crosses the x -axis (since we want $y=0$)

Simultaneous Equations

The equation of a straight line.

Any linear equation in two variables may be represented graphically by plotting the values of one variable against the values of the other. The gradient of such a line is defined to be the vertical distance between any two points divided by the horizontal distance and is usaully denoted by m so if the variables are the usual x and y we have $m=\dfrac{y_2-y_1}{x_2-x_1}$ note that the suffixes must be in the same order on top and bottom.

(a) Given the gradient m and the coordinates $(h,\ k)$ of a point on the line then if (x,y) is any other point on the line we must have $m=\dfrac{y-k}{x-h}\Rightarrow y-k=m(x-h)$ and the equation of the line is thus $y-k=m(x-h)$

(b) Given any two points on the line we would generally find the gradient and use the previous result with either of the given points as (h,k)

(c) Rearranging $y-k=m(x-h)$ we have $y=mx-mh+k$ or $y=mx+c$

This is one of the standard forms for the equation. Note that obviously, c is the value of y at $x = 0$, i.e. the intercept on the y axis. This is therefore known as the GRADIENT - INTERCEPT form of the equation.

(d) The other most common form, usually referred to as the GENERAL LINEAR EQUATION is obtained by rearranging again into the form $ax+by+c=0$

Note carefully that the values of c in the above two general forms are NOT the same.

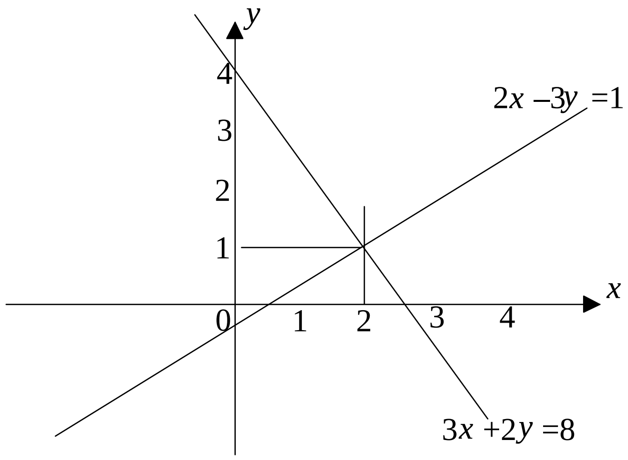
Intersection of two lines or curves.

The points of intersection of two lines or curves is found by solving their equations simultaneously.

(1) Two straight lines

.There are three standard methods for solving a pair of linear simultaneous equations, ie finding the values of x and y which satisfy both equations at the same time.
(1) Graphically. Since a linear equation in x and y can be represented by a straight line on a graph it follows that the simultaneous solution is given by the point where the two lines intersect.
Ex. Find the simultaneous solution of $2x-3y=1$ and $3x+2y=8$

To draw the two lines the simplest way is to find where they intersect the axes. So for $2x-3y=1$ we have, $x=0 \Rightarrow y=-\frac{1}{3}$ and $y=0 \Rightarrow x=\frac{1}{2}$ For $3x+2y=8$, $x=0 \Rightarrow y=4$ and $y=0 \Rightarrow x=\frac{8}{3}$ Note if a line passes through the origin then take some other value of x or y to obtain a second point. We now draw the lines on the same axes and read off the point of intersection.



From our graph we can see that the point of intersection and therefore the simultaneous solution of the equations is at $x=2,\ y=1$
Solving graphically is however, at best only approximate and so we consider more analytical methods.

(2) By substitution. This method involves expressing one unknown in terms of the other from one equation and then substituting that in the other equation.
Ex. Using the same example as before, from $2x-3y=1$ we have $x=\frac{1}{2}(3y+1)$ so substituting for x in $3x+2y=8$ gives $\frac{3}{2}(3y+1)+2y=8 \Rightarrow 9y+3+4y=16 \Rightarrow 12y=12 \Rightarrow y=1$
substituting back in the first equation we then have $2x-3=1 \Rightarrow x=2$ so the solution is as before.

(3) By Elimination The final method involves arranging for either the x terms or the y terms in the two equations to have the same numerical coefficients and then adding or subtracting the equations to eliminate that particular term.
Ex. Again from $2x-3y=1$ and $3x+2y=8$, multiplying the first by 2 and the second by 3 gives $4x-6y=2$ and $9x+6y=24$ Adding the left hand sides and right hand sides eliminates y and gives $13x=26 \Rightarrow x=2$. Substituting back in either of the equations will then give $y=1$ as before.
Note that if the two terms to be eliminated have the same sign then you must subtract one equation from the other. Whenever possible, choose to eliminate the one that you can by adding rather than subtracting as there is less likelihood of making a mistake. Note, as a general rule, if the substitution method is going to involve fractions then the elimination method is to be preferred.

(2) A straight line and a curve
Now we only have two possible methods, graphical, which of course can only give approximate solutions or substitution, which we illustrate.
Ex. Solve simultaneously $x^2+y^2-4x+6y=0$ and $y=2x-3$
We substitute from the linear into the quadratic thus : $x^2+(2x-3)^2-4x+6(2x-3)=0$
ie $x^2+4x^2-12x+9-4x+12x-18=0 \Rightarrow 5x^2-4x-9=0 \Rightarrow (5x-9)(x+1)=0$
So $x=\frac{9}{5}$ or -1. Substituting back into the linear equation, $x=\frac{9}{5} \Rightarrow y=\frac{3}{5}$ and $x=-1 \Rightarrow y=-5$
The solutions are thus $x=-1,\ y=-5$ and $x=\frac{9}{5},\ y=\frac{3}{5}$
Ex. Solve $x-3y=2$ and $x^2-2xy-3y^2=12$
This can be done by the normal substitution method or more simply by noticing that $x-3y$ is a factor of $x^2-2xy-3y^2$ so our equations may be written as $x-3y=2$ and $(x-3y)(x+y)=12$ so substituting for $x-3y$ in the second equation gives $2(x+y)=12$ or $x+y=6$ and we have reduced the problem to a pair of linear equations which are easily solved to give $4y=4 \Rightarrow y=1, x=5$

Changing the subject of a formula.

This is very similar to solving an equation, the object being to rearrange the formula so that the variable you are interested in is on its own on the left hand side.
Ex. Make u the subject of $v^2=u^2+2fs$
1.It is convenient to first write the formula in reverse to avoid awkward negative signs later so we start with $u^2+2fs=v^2$ and subtract the $2fs$ term from both sides to leave u^2 on its own, giving

$u^2 = v^2 - 2fs$

2. Now all we have to do to have u on its own is to take the square root of each side $u = \pm\sqrt{v^2 - 2fs}$

If you know that u cannot be negative you can omit the $\pm$ sign.

If the required variable appears in more than one part of the formula you have to be more careful.

Ex. Make x the subject of the formula $3xy - 2 = y^2 + 4x - 7$

1. The first step is to collect together on one side of the equation all the terms involving the required subject and no other terms so subtracting $4x$ and adding 2 to both sides we have $3xy - 4x = y^2 - 5$

2. Now we just need to factorise the left hand side to give $x(3y-4) = y^2 - 5$ and divide both sides by the factor $3y - 4$ to give $x = \frac{y^2-5}{3y-4}$

Inequalities.

Most of the following results should already be familiar

Properties of > and < $\quad x > y \Rightarrow x + a > y + a$ and $x - a > y - a$ for all real a

$ax > ay$ and $\frac{x}{a} > \frac{y}{a}$ for $a > 0$

But $\quad ax < ay$ and $\frac{x}{a} < \frac{y}{a}$ for $a < 0$

Note particularly these last two properties.

Powers and roots must be handled with care and it is safer to reason carefully each time rather than try to rely on memorised results.

Ex. $a > b \Rightarrow a^2 > b^2$ only if a is numerically greater than b. Otherwise for example $2 > -4$ but $4 < 16$

$x^2 > a \Rightarrow x > \sqrt{a}$ or $x < -\sqrt{a}$ and $x^2 < a \Rightarrow -\sqrt{a} < x < \sqrt{a}$

Solving inequalities

This is done in exactly the same way as for equations but being careful to pay regard to the above properties.

Ex. $7 - 2x > 4 \Rightarrow 3 > 2x \Rightarrow x < \frac{3}{2}$

Ex. $(x-3)(x+2) > 0$ Here we can think of $x = 3$ and $x = -2$ as 'critical values' of x since it is only as x passes through either of these values that the expression can change sign.

If $x < -2$ then $x - 3$ and $x + 2$ are both negative and so their product is positive and the inequality is satisfied.

If $-2 < x < 3$ then $x - 3 < 0$ but $x + 2 > 0$ so the product is negative and the inequality is not satisfied.

If $x > 3$ then $x - 3$ and $x + 2$ are both positive and the inequality is satisfied again.

Thus the inequality holds for $x < -2$ and for $x > 3$. Another approach here is to think of the graph of $y = (x-3)(x+2)$ which has a positive x^2 term and so will be 'hanging down' which necessarily means that it is negative for values of x between the roots of the equation $(x-3)(x+2) = 0$

Modulus inequalities.

The modulus of a real number x, written $|x|$ is the numerical value of x. ie $|5| = 5$, $|-3| = 3$. Formally, $|x| = x$ if $x \geqslant 0$ but $|x| = -x$ if $x < 0$ Thus the modulus function can never take a negative value.

$|x| < a \Rightarrow -a < x < a$, $|x| > a \Rightarrow x < -a$ or $x > a$

Ex. $|2x+1| < 3 \Rightarrow -3 < 2x + 1 < 3 \Rightarrow -4 < 2x < 2 \Rightarrow -2 < x < 1$

Ex. $|2x-1| < |x+3|$ With modulus signs on both sides of the inequality it is best to proceed as follows:

Since both sides are positive then $[2(x-1)]^2 < (x+3)^2 \Rightarrow 4x^2 - 8x + 4 < x^2 + 6x + 9$

Hence, $3x^2 - 14x - 5 < 0 \Rightarrow (3x+1)(x-5) < 0 \Rightarrow -\frac{1}{3} < x < 5$

Quadratic Inequalities

These are best demonstrated by examples.

Algebraic solutions

Ex. Solve the inequality $x^2 - 7x + 6 < 12 - 2x$

First rearrange to have zero on one side i.e. $x^2 - 5x - 6 < 0$ then factorise if possible

$(x-6)(x+1) < 0$ Now consider the values which satisfy $x^2 - 5x - 6 = 0$ i.e. $x = 6$ or -1

Since the expression can only possibly change sign at these two values we need only consider the three ranges of values $x < -1$, $-1 < x < 6$ and $x > 6$

Clearly, if $x < -1$ then $x - 6 < 0$ and $x + 1 < 0$ so $(x-6)(x+1) > 0$ and the inequality is not satisfied.

If $-1 < x < 6$ then $x - 6 < 0$ and $x + 1 > 0$ so $(x-6)(x+1) < 0$ and inequality is satisfied

If $x > 6$ then both factors are positive and the inequality is satisfied so solution is $-1 < x < 6$

Ex. Solve the inequality $\frac{1}{x-1} > \frac{x}{3-x}$

Beware of simply multiplying both sides by $(x-1)$ and $(3-x)$ since you have no way of knowing whether these are positive or negative numbers. Instead proceed as follows:

Get zero on one side as before $\frac{1}{x-1} - \frac{x}{3-x} > 0$ and put over a common denominator

$\frac{3-x-x(x-1)}{(x-1)(3-x)} > 0 \Rightarrow \frac{3-x^2}{(x-1)(3-x)} > 0$ and consider all values which make either the numerator or denominator zero i.e. $x = 1,\ 3$ or $\pm\sqrt{3}$

This gives us the ranges of values $x < \sqrt{3}, -\sqrt{3} < x < 1,\ 1 < x < \sqrt{3},\ \sqrt{3} < x < 3$ and $x > 3$ to consider.

The reasoning is best set down in a table as follows:

	$x < -\sqrt{3}$	$-\sqrt{3} < x < 1$	$1 < x < \sqrt{3}$	$\sqrt{3} < x < 3$	$x > 3$
$3 - x^2$	–	+	+	–	–
$x - 1$	–	–	+	+	+
$3 - x$	+	+	+	+	–
	+	–	+	–	+

the table shows that $\frac{3-x^2}{(x-1)(3-x)} > 0$ for $x < -\sqrt{3},\ 1 < x < \sqrt{3}$ and $x > 3$

Such inequalities may also be solved approximately by a graphical method. i.e. plot the graphs of $y = \frac{1}{x-1}$ and $y = \frac{x}{3-x}$ and see where the graph of $y = \frac{1}{x-1}$ is above that of $y = \frac{x}{3-x}$

Surds

A SURD is an expression involving only rational numbers and their roots. (A RATIONAL number is one which can be written in the form $\frac{a}{b}$ where *a* and *b* are integers (whole numbers))

Eg $\sqrt{2}$, $\sqrt{\frac{3}{5}}$, $(\sqrt{2} + \frac{1}{\sqrt{3}-1})$, $3 + \sqrt{2} + \sqrt[3]{5}$ etc.

Note! If $x > 0$, then $\sqrt[m]{x} = x^{\frac{1}{m}}$ means the positive m^{th} root of *x*. i.e. $\sqrt{9} = 3$ and NOT ±3

Simplification of surds.

The following results are given for quadratic surds only but apply in general to other surds.

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ (iii) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$

Ex. Simplify $\sqrt{128} - 3\sqrt{32} + \sqrt{8} + 3\sqrt{2}$

$\sqrt{128} = \sqrt{64 \times 2} = \sqrt{64}\sqrt{2} = 8\sqrt{2}$ similarly $\sqrt{32} = 4\sqrt{2}$ and $\sqrt{8} = 2\sqrt{2}$

So $\sqrt{128} - 3\sqrt{32} + \sqrt{8} + 3\sqrt{2} = 8\sqrt{2} - 12\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} = \sqrt{2}$

In general, if *n* has a perfect square factor then $\sqrt{n}$ should be simplified., so you should not, in general, leave answers as $\sqrt{50}$ or $\sqrt{27}$ but should give them as $5\sqrt{2}$ and $3\sqrt{3}$ respectively.

Rationalising a denominator. This means, express a fraction with a surd denominator in a form such that the denominator is not a surd.

Ex. $\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{5}$

Ex. $\frac{1}{\sqrt{5}+\sqrt{2}} = \frac{1}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{\sqrt{5}-\sqrt{2}}{5-2} = \frac{\sqrt{5}-\sqrt{2}}{3}$

Ex. $\frac{\sqrt{2}-1}{\sqrt{3}-2} = \frac{\sqrt{2}-1}{\sqrt{3}-2} \times \frac{\sqrt{3}+2}{\sqrt{3}+2} = \frac{\sqrt{6}-\sqrt{3}+2\sqrt{2}-2}{3-4} = 2 - 2\sqrt{2} + \sqrt{3} - \sqrt{6}$

Indices.

You need to be very fluent in the use and manipulation of indices. The main laws are as follows

(a) A positive whole number index tells us how many of that item are to be multiplied together.

Ex. x^4 means $x \times x \times x \times x$. $3x^2y^3$ means $3 \times x \times x \times y \times y \times y$

(b) $x^a \times x^b = x^{a+b}$ eg $x^6 \times x^4 = x^{10}$

(c) $x^a \div x^b = x^{a-b}$ eg $\frac{x^5}{x^2} = x^5 \div x^2 = x^3$

(d) $[x^a]^b = x^{ab}$ eg $\left(x^3\right)^4 = x^{12}$

(e) The meaning of a negative index can be deduced as follows:

$x^{-3} = x^2 \div x^5 = \frac{x^2}{x^5} = \frac{1}{x^3}$ by dividing top and bottom by x^2 generalising $x^{-a} = \frac{1}{x^a}$

(f) From (c) it follows that $x^0 = x^a \div x^a = 1$ for ALL values of x

(g) The meaning of a fractional index can now be deduced:

Since $\left[x^{1/2}\right]^2 = x^1 = x$ it follows that $x^{1/2} = \sqrt{x}$ and in general, $x^{1/n}$ is the n^{th} root of x

It then follows that $x^{a/b}$ is the b^{th} root of x raised to the power of a or the b^{th} root of x to the power of a

<u>Ex</u>. $32^{\frac{3}{5}} = \left[32^{\frac{1}{5}}\right]^3 = 2^3 = 8$ note that it is usually best to work out the root first and then the power though the answer is the same if you do it the other way round.

Ex. Simplify $8^{2/3} \times 27^{-4/3} \times 2^{-2} \times 13^0$

$8^{2/3} = 2^2 = 4,\ 27^{-4/3} = \frac{1}{27^{\frac{4}{3}}} = \frac{1}{3^4} = \frac{1}{81},\ 2^{-2} = \frac{1}{2^2} = \frac{1}{4},\ 13^0 = 1$

Hence $8^{2/3} \times 27^{-4/3} \times 2^{-2} \times 13^0 = 4 \times \frac{1}{81} \times \frac{1}{4} = \frac{1}{81}$

<u>Ex</u>. Solve the equation $3^x.2^{2x-3} = 18$

The secret here is to express everything in powers of 2 and 3 and collect terms together.

We have $3^x.\frac{2^{2x}}{2^3} = 2.3^2 \Rightarrow 3^x.2^{2x} = 2^4.3^2$ and by inspection we can see that x must be equal to 2.

<u>Co-ordinate Geometry</u>

<u>Elementary Coordinate Geometry</u>

It is assumed that you are familiar with the method of defining points in a plane by means of Cartesian coordinates, i.e. ordered pairs of numbers $(x,\ y)$ with respect to fixed perpendicular axes in the plane Ox and Oy.

You have probably met most of the following results before but they are very important and we give them again for the sake of completeness.

<u>Distance between two points and coordinates of midpoint.</u>

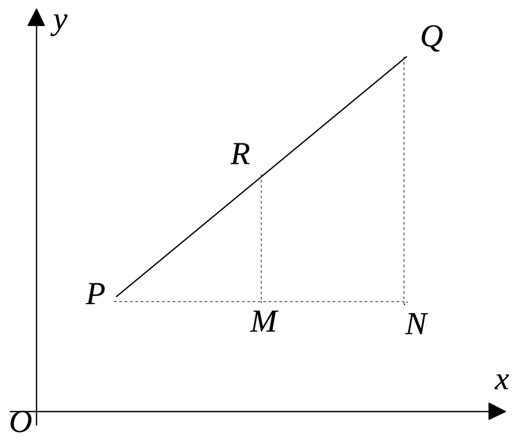
Consider the points $P(x_1, y_1)$ and $Q(x_2, y_2)$

By Pythagoras in ΔPNQ

$PN = x_2 - x_1$ and $NQ = y_2 - y_1$ hence

$PQ^2 = PN^2 + NQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\Rightarrow PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



If $R(\alpha, \beta)$ is the midpoint of PQ then since

Δs PQN, PRM are similar, $PM = \frac{1}{2}PN$

ie $\alpha - x_1 = \frac{1}{2}(x_2 - x_1) \Rightarrow \alpha = \frac{1}{2}(x_1 + x_2)$

Similarly, $\beta = \frac{1}{2}(y_1 + y_2)$ and so we see that the coordinates of the midpoint are the average of the coordinates of the ends.

<u>Gradient of a line</u>

<u>Parallel and perpendicular lines</u>

The GRADIENT (or SLOPE) of $PQ = \frac{NQ}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$

It should be obvious that if two lines are parallel then they have the same gradient, and vice versa.

In the diagram, OP and OP' are perpendicular

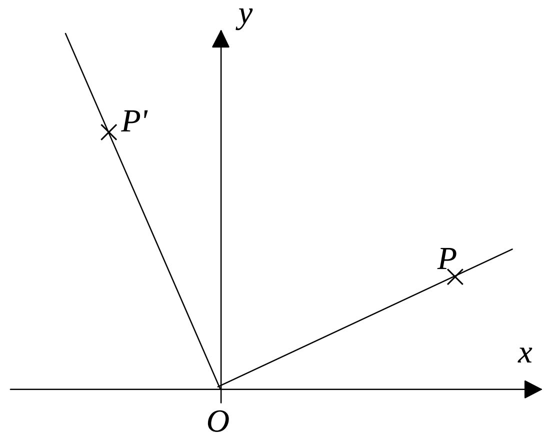
Let P be the point $(h,\ k)$ then, if P' is the image of P under an anti-clockwise rotation of 90° about O then P' is the point $(-k,\ h)$

Gradient of $OP = \frac{k}{h}$ and gradient of $OP' = \frac{h}{-k} = -\frac{h}{k}$

So we see that the product of the gradients is -1

Thus, lines with gradients m_1, m_2 are parallel if

And only if $m_1 = m_2$ and perpendicular if, and only if $m_1 m_2 = -1$



<u>The equation of a straight line</u>.

The equation of a line (or curve) is defined to be the relationship between the coordinates of an arbitrary point on the line (or curve). There are various standard forms for the equation of a straight line often depending on the information given or required.

(a) Given the gradient m and the coordinates $(h,\ k)$ of a point on the line then if (x, y) is any other point on the line we must have $m = \frac{x-k}{y-h} \Rightarrow y - k = m(x - h)$ and the equation of the line is thus $y - k = m(x - h)$

(b) Given any two points on the line we would generally find the gradient and use the previous result with either of the given points as (h, k)

(c) Rearranging $y - k = m(x - h)$ we have $y = mx - mh + k$ or $y = mx + c$

This is one of the standard forms for the equation. Note that obviously, c is the value of y at $x = 0$, i.e. the intercept on the y axis. This is therefore known as the GRADIENT - INTERCEPT form of the equation.

(d) The other most common form, usually referred to as the GENERAL LINEAR EQUATION is obtained by rearranging again into the form $ax + by + c = 0$

Note carefully that the values of c in the above two general forms are NOT the same.

<u>Equation of a Circle</u>

Consider a circle, centre at (h, k) with radius r. If (x, y) is any point on the circumference then its distance from the centre is r and so by Pythagoras we have $(x-h)^2 + (y-k)^2 = r^2$ which is thus the equation of the circle. Multiplying out we have $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$ and so the general equation of a circle may be written in the form

$$x^2 + y^2 + 2fx + 2gy + c = 0 \text{ with centre at } (-f, -g) \text{ and radius } \sqrt{f^2 + g^2 - c}$$

Note in particular that the equation of a circle has equal coefficients of x^2 and y^2 and no xy term.

In general we can easily find the centre and radius of a circle by completing the square of the x and y terms and comparing with the first form above.

<u>Common circle properties.</u>

1. The angle in a semicircle is a right angle.
2. The perpendicular from the centre of a circle to a chord bisects the chord.
3. The tangent to a circle at a point is perpendicular to the radius through that point

Polynomials

Basic work on polynomials was covered in the Algebra section.

The Factor and Remainder Theorems

Given a polynomial in x of degree n we can, in general, divide this by a linear expression $x-k$ to give a quotient $\mathrm{Q}(x)$ which will be a polynomial in x of degree $n-1$ together with a numerical remainder. Thus $\mathrm{P}(x)=(x-k)\,\mathrm{Q}(x)+R$ and putting $x=k$ we see that $\mathrm{P}(k)=R$. This is the REMAINDER THEOREM which states "When a polynomial in x is divided by $x-k$ then the remainder is the value of $\mathrm{P}(k)$"

Ex. Find the remainder when $\mathrm{P}(x)=3x^5-2x^4-x^3+x^2-5x+6$ is divided by $x-2$

Remainder is $\mathrm{P}(2)=96-32-8+4-10+6=56$

You should verify this result by long division.

A special case arises when $x-k$ is a factor of $\mathrm{P}(x)$ for the division is then exact and we have $R=0$

So $\mathrm{P}(k)=0$ and the converse is also true. We thus have the FACTOR THEOREM which states "If $\mathrm{P}(k)=0$ then $x-k$ is a factor of $\mathrm{P}(x)$ and conversely" This is of great value in solving equations of degree greater than 2.

Ex. Solve the equation $x^5-x^4-4x^3-4x^2-5x-3=0$

First we note that any linear factor must be of the form $x\pm1$ or $x\pm3$ (why?)

Let $\mathrm{P}(x)=x^5-x^4-4x^3-4x^2-5x-3$ then $\mathrm{P}(1)=1-1-4-4-5-3\neq0$ so $x-1$ is not a factor.

$\mathrm{P}(-1)=-1-1+4-4+5-3=0$ so $x+1$ is a factor and so

$\mathrm{P}(x)=(x+1)(x^4-2x^3-2x^2-2x-3)$ so let $x^4-2x^3-2x^2-2x-3=\mathrm{Q}(x)$

Note that when we continue we must start again with $x+1$ in case it is a repeated factor

$\mathrm{Q}(-1)=1+2-2+2-3=0$ so $\mathrm{Q}(x)=(x+1)(x^3-3x^2+x-3)$ and let $x^3-3x^2+x-3=\mathrm{S}(x)$

$\mathrm{S}(-1)=-1-3-1-3\neq0,\ \mathrm{S}(3)=27-27+3-3=0$ so $\mathrm{S}(x)=(x-3)(x^2+1)$

And so we have $\mathrm{P}(x)=(x+1)^2(x-3)(x^2+1)=0\Rightarrow x=-1$ (repeated) or 3

Ex. The polynomial $2x^3+ax^2+bx+6$ is exactly divisible by $x-2$ and leaves a remainder of -12 when divided by $x+1$, find the values of a and b

Let $\mathrm{P}(x)=2x^3+ax^2+bx+6$ then $\mathrm{P}(2)=0$ and $\mathrm{P}(-1)=-12$

i.e. $16+4a+2b=6=0$ and $-2+a-b+6=0$ or $2a+b=-5$ and $a-b=-4\Rightarrow a=-3,\ b=1$

Graphs of Polynomial Functions

By factorising a polynomial we can determine where it intersects the x-axis, i.e. values of x for which the polynomial has a zero value. Consideration of the sign between any two such values enables us to sketch the graph.

The Binomial Expansion

By direct multiplication we find that

$$\begin{aligned}(1+x)^1&=1+x\\(1+x)^2&=1+2x+x^2\\(1+x)^3&=1+3x+3x^2+x^3\\(1+x)^4&=1+4x+6x^2+4x^3+x^4\end{aligned}$$

Can you see how each row is obtained from the previous row, see how we obtain $(1+x)^5$

$(1+x)^5=(1+x)(1+x)^4=(1+x)\left(1+4x+6x^2+4x^3+x^4\right)$

Now $1\times\left(1+4x+6x^2+4x^3+x^4\right)=1+4x+6x^2+4x^3+x^4$ and

$x\times\left(1+4x+6x^2+4x^3+x^4\right)=\quad x+4x^2+6x^3+4x^4+x^5$ so adding we get

$$(1+x)^5=1+5x+10x^2+10x^3+5x^4+x^5$$

So we can see that each coefficient is the sum of the corresponding coefficient in the previous expansion added to the one on its left. Continuing this pattern produces PASCAL's triangle, (named after a French Mathematician Blaise Pascal (1623-1642))

1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

By extending the pattern for another three rows, write down the expansion of $(1+x)^8$

Note the following features

(i) the number of terms in the expansion of $(1+x)^n$ is $n+1$

(ii) the coefficients are symmetrical.

Can we find coefficients without having to write out Pascal's triangle? Consider the term in x^r in the expansion of $(1+x)^n$. The r - x's can be taken from any r of the n factors of 1+ x and so the number of such terms and therefore the coefficient of x^r is the number of ways of selecting the r factors which are to contribute the x's. We write this as ${}^n\mathrm{C}_r$ or $\binom{n}{r}$ Thus

$$(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3++\binom{n}{r}x+\ldots+x^n$$

This is the BINOMIAL THEOREM for positive integral indices.

If you already know how to calculate ${}^n\mathrm{C}_r$ then you can skip the following explanation.

To evaluate ${}^n\mathrm{C}_r$ we note that selecting r things from n things is just the same as putting one thing in each of r boxes when we have n things to choose from. We argue as follows:

The first box can be filled in n different ways, the second in $n-1$ ways (since one of the n objects has already been used), the third in $n-2$ ways and so on until all the boxes have been filled. This would seem to give a total number of selections of $n(n-1)(n-2)\ldots(n-r+1)$ i.e r factors. However this takes account of the order in which the boxes are filled and clearly, the order in which the x's are selected does not matter. Now r things can be arranged amongst themselves in $r(r-1)(r-2)\ldots.3.2.1 = r!$ (r-factorial or factorial-r) ways by a similar argument to that above. Thus, each basically different selection is repeated $r!$ times and so we have

$${}^n\mathrm{C}_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \text{ or } \frac{n!}{r!(n-r)!}$$

Note particularly the pattern of r factors in the numerator and r-factorial in the denominator.

Example (i) Calculate ${}^{10}\mathrm{C}_4$

$${}^{10}\mathrm{C}_4 = \frac{10.9.8.7}{4!} = \frac{10.9.8.7}{4.3.2.1} = \underline{210}$$

Example (ii) Obtain the first three terms in the expansion of $(1-2x)^8$

$$(1-2x)^8 = 1+8(-2x)+\frac{8.7}{2}(-2x)^2.. = 1-16x+112x^2$$

Example (iii) Find the first four terms in the expansion of $(3+2x)^6$

Note especially how we deal with this problem.

$$(3+2x)^6 = 3^6\left(1+\tfrac{2}{3}x\right)^6 = 3^6\left(1+6\left(\tfrac{2}{3}x\right)+\frac{6.5}{2}\left(\tfrac{2}{3}x\right)^2+\frac{6.5.4}{3.2.1}\left(\tfrac{2}{3}x\right)^3..\right) = 3^6\left(1+4x+20x^2+\frac{160}{27}x^3\right)$$

$$= \underline{729+2916x+14580x^2+4320x^3}$$

Example (iv) Find the term in x^6 in the expansion of $(2-3x^2)^5$

We obviously require the term in $(x^2)^3$

First we must write $(2-3x^2)^5 = 2^5\left(1-\frac{3}{2}x^2\right)^5$ so the required term is

$$2^5 \times \frac{5.4.3}{3!}\left[\frac{3}{2}x^2\right]^3 = 320\times\frac{27}{8}x^6 = \underline{1080x^6}$$

The binomial theorem can also be used to calculate certain values, thus

Example (v) Find, without a calculator, the value of $(0.997)^{10}$ giving your answer correct to 4 significant figures.

$(0.997)^{10} = (1-x)^{10}$ where $x = 0.003$ Now $(1-x)^{10} = 1-10x+45x^2-120x^3+\ldots$ so, putting $x = 0.003$ we have $(0.997)^{10} = 1-0.03+45\times0.000009-120\times0.000000027$

$= 0.97+0.000405$ the next term clearly will not affect the 4th sig.fig. Hence, $\underline{(0.997)^{10} = 0.9704}$

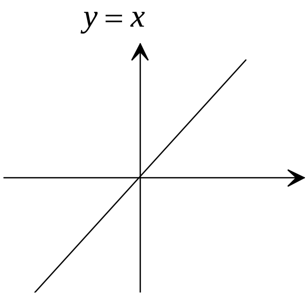
Curve Sketching

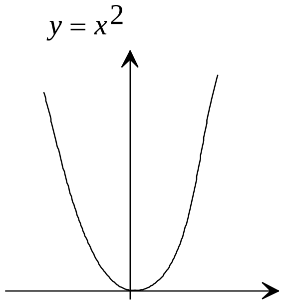
When asked to sketch a curve you should not be attempting to plot points accurately, rather just showing the essential features, i.e. approximate turning points, intersection with axes etc.

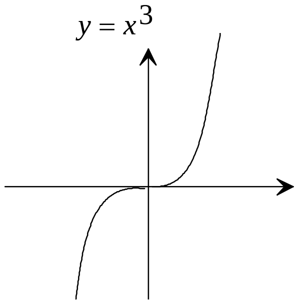
As an initial basis be sure you know the following standard curves.

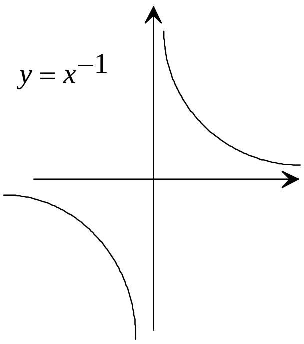
Curves in general

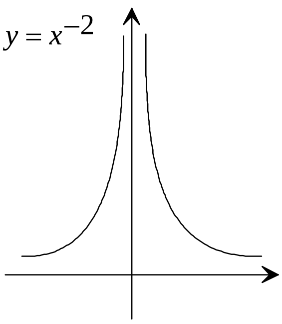
You should learn the shapes of certain standard curves and be able to sketch their graphs without having to plot them point by point. The following are the most common











In general, for positive values of n $y = x^n$ will have a graph of the form of $y = x^2$ if n is even and of the form $y = x^3$ if n is odd. Similarly, for negative n we will have a graph of the form $y = x^{-1}$ or $y = x^{-2}$ according as$n$ is odd or even.

Transformations

You should be familiar with the common transformations from your GCSE work but we consider here the effect that they have on the graph of a function.

(1) The transformation of $y = \mathrm{f}(x)$ represented by $y = a\mathrm{f}(x)$ Clearly for each given value of x, this transformation multiplies the corresponding value of the function by a. Graphically this 'stretches' the curve parallel to the y - axis by a factor of 'a', points on the x - axis remaining fixed.

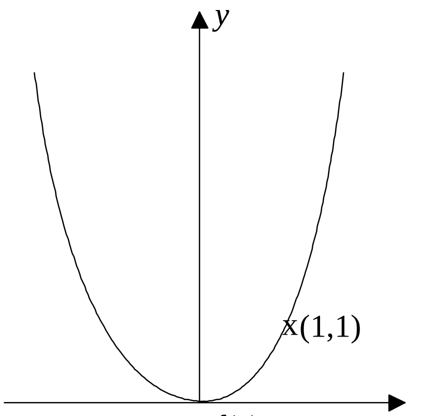
(2) $y = \mathrm{f}(x) + a$. This clearly adds a to each function value and so is a translation of magnitude a parallel to the y - axis.

(3) $y = \mathrm{f}(x - a)$ is a translation of magnitude a parallel to the x - axis

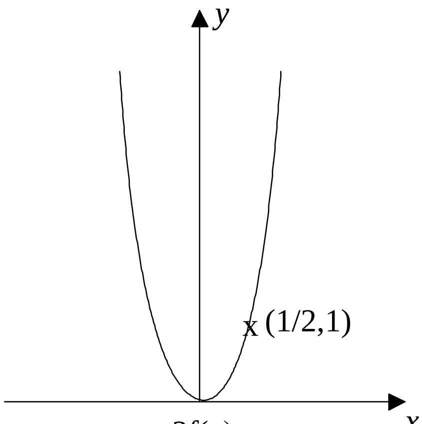
(4) $y = \mathrm{f}(ax)$ is a stretch parallel to the x axis by a factor of $\frac{1}{a}$

Ex. Show the effects on the curve $y = x^2$ of the transformations (a) $y = 2\mathrm{f}(x)$ (b) $y = \mathrm{f}(x) + 2$
(c) $y = \mathrm{f}(x - 2)$ (d) $y = \mathrm{f}(2x)$ and in each case show the image of the point $P(1, 1)$

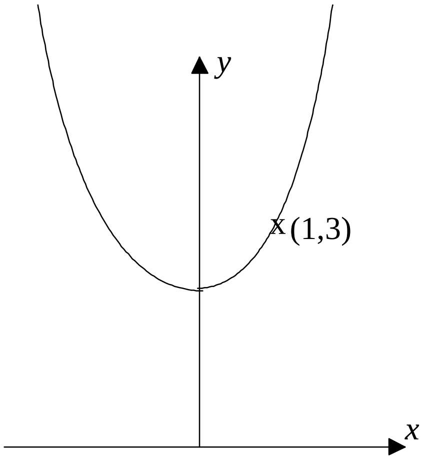
Solution on next page



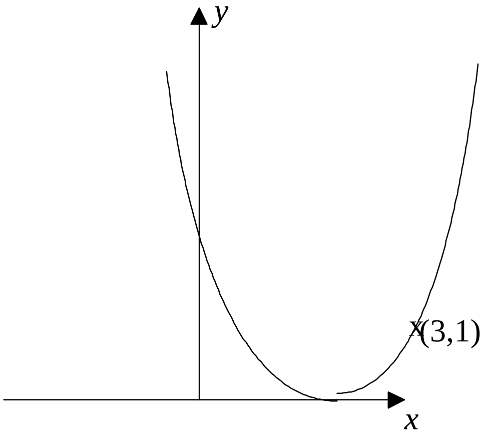
$y = \mathrm{f}(x)$



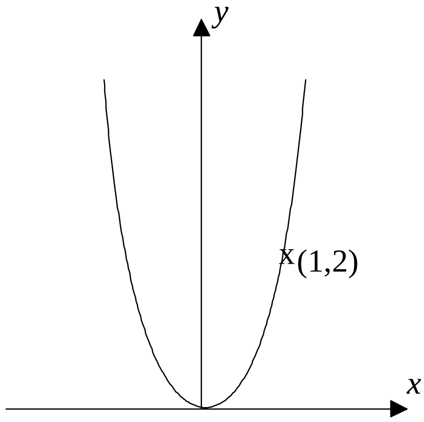
$y = 2\mathrm{f}(x)$



$y = \mathrm{f}(x) + 2$



$y = \mathrm{f}(x-2)$



$y = \mathrm{f}(2x)$

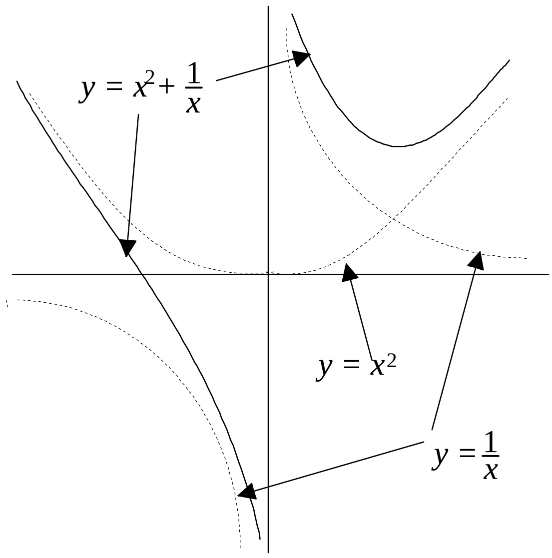
<u>General Curve Sketching</u>

It is very useful to be able to produce a quick freehand sketch of the graph of a function, showing its essential features, without the labour of working out and accurately plotting points. The important features to consider are (i) symmetry, (ii) intersections with axes, (iii) discontinuities and asymptotes, (iv) turning points and (v) behaviour as $x \to \pm\infty$

The first and most important rule is that you should always use the simplest methods possible. Many graphs can be sketched by using the knowledge gained in your pre-A level , of the graphs of $y = x^n$ for $n = \pm 1, \pm 2, \pm 3$

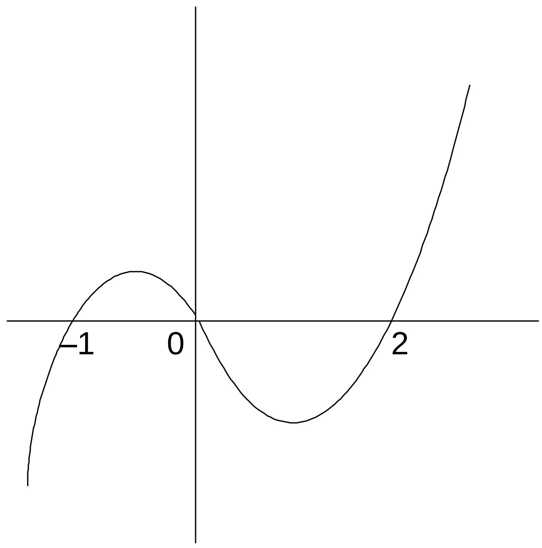
Ex. Sketch the graph of $y = x^2 + \frac{1}{x}$

Here we can simply sketch the graphs of $y = x^2$ and $y = \frac{1}{x}$ and 'add them'



Ex. Sketch the graph of $y = x^3 - x^2 - 2x$

Here we have a cubic and factorising we see that $y = x(x+1)(x-2)$ and so the curve crosses the x acxis at $(-1,0)$, $(0,0)$ and $(2,0)$ Clearly, $y > 0$ for large values of x and $y < 0$ for $0 < x < 2$ so, knowing the fundamental shape of a cubic we can easily sketch the graph.

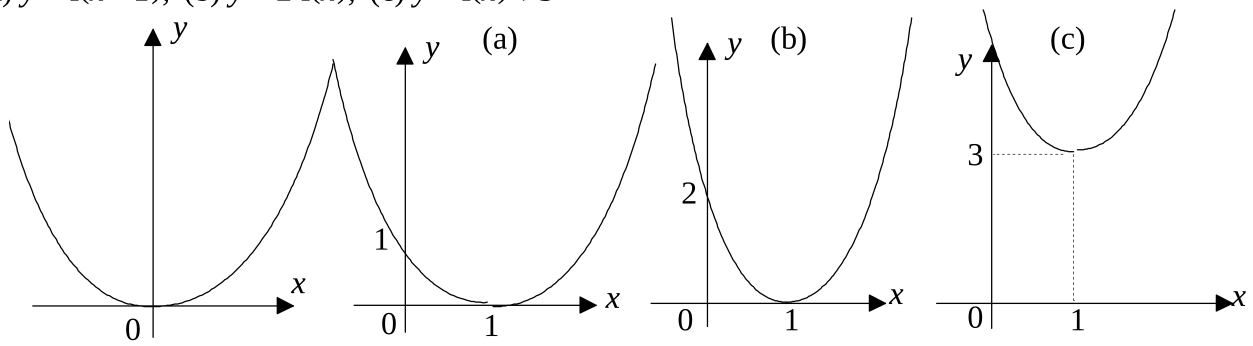


Another way of sketching graphs is to make use of the transformations described previously.

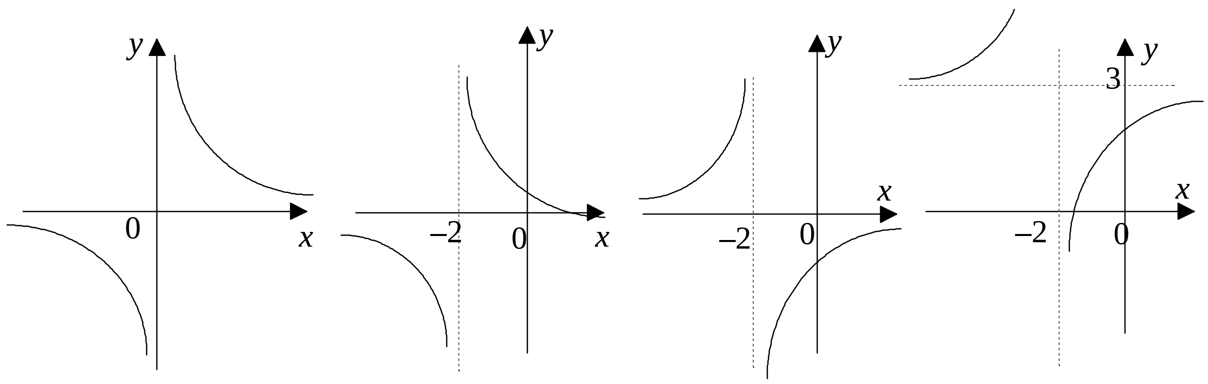
Ex. Use transformations to sketch the graph of (i) $y = 2x^2 - 4x + 5$ (ii) $y = \frac{3x+5}{x+2}$

Solution. (i) We can write the equation as $y = 2(x-1)^2 + 3$ by completing the square

Starting from $y = x^2$ the required equation is obtained by applying, in succession, the transformations (a) $y = \mathrm{f}(x-1)$, (b) $y = 2\,\mathrm{f}(x)$, (c) $y = \mathrm{f}(x) + 3$



(ii) $y = \frac{3x+5}{x+2} = 3 - \frac{1}{x+2}$ so starting from $y = \frac{1}{x}$ we apply the transformations (a) $y = \mathrm{f}(x+2)$, (b) $y = -\,\mathrm{f}(x)$ and (c) $y = \mathrm{f}(x) + 3$



(a) (b) (c)

Note that the transformation $y = -\mathrm{f}(x)$ effects a reflection in the x –axis

Algebra

Logarithms

Before the invention of the calculator, the main way of performing complex calculations was to use tables of logarithms. Logarithms are simply powers of some fixed number, known as the BASE of the logarithms. For simple calculations, the most suitable base is 10 and such logarithms are called COMMON LOGARITHMS. It is quite easy to convert numbers into approximate powers of 10 by drawing a graph of $y = 10^x$ for values of x from 0 to 1. A sufficient number of points can be plotted using square root tables and the laws of indices.

Ex. $10^{0.5} = \sqrt{10} = 3.162$, $10^{0.25} = \sqrt{3.162} = 1.778$, $10^{0.75} = 10^{0.25} \times 10^{0.5} = 3.162 \times 1.778$

i.e $10^{0.75} = 5.622$. More can be obtained in a similar way if required and a smooth curve drawn through the points. Reading the graph in reverse you can obtain approximate values for the logarithm of any number between 1 and 10. Fortunately, tables giving these values correct to 4 significant figures are readily available and so calculations can be done quite accurately. Final answers usually being reliable to 3 significant figures. Once we can convert numbers into powers of 10 it is easy to perform various calculations.

Ex. (i) Calculate (a) 32.7×0.00764 (b) $9.64 \div 23.8$ (c) 4.29^4 (d) $\sqrt[3]{172}$

(a) $32.7 \times 0.00764 = (3.27 \times 10) \times \left(7.64 \times 10^{-3}\right)$ Note that we have to convert the numbers to standard form (i.e $k \times 10^n$ where $0 < k < 1$ and n is an integer) because the tables only give logarithms of numbers between 1 and 10. Continuing

$32.7 \times 0.00764 = 3.27 \times 7.64 \times 10^{-2} = 10^{0.5145} \times 10^{0.8831} \times 10^{-2} = 10^{1.3976} \times 10^{-2}$

$10^{0.3976} \times 10^{-1} = 2.498 \times 10^{-1}$ (by using the table in reverse)= $\underline{0.250}$

(b) $9.64 \div 23.8 = 9.64 \div (2.38 \times 10) = 10^{0.9841} \div (10^{0.3766} \times 10) = 10^{0.9841} \div 10^{1.3766}$

$= 10^{-0.3925} = 10^{0.6075} \times 10^{-1} = 4.050 \times 10^{-1} = \underline{0.405}$

Note in particular the step from $10^{-0.3925}$ to $10^{0.6075} \times 10^{-1}$

(c) $4.29^4 = (10^{0.6325})^4 = 10^{0.6325 \times 4} = 10^{2.53} = 10^{0.53} \times 10^2 = 3.388 \times 10^2 = \underline{339}$

(d) $\sqrt[3]{172} = (1.72 \times 10^2)^{\frac{1}{3}} = \left(10^{0.2355} \times 10^2\right)^{\frac{1}{3}} = 10^{2.2355 \div 3} = 10^{0.7452} = \underline{5.56}$

These examples illustrate the complete process and how we make use of the laws of indices with which you should already be familiar, but in practice we set the working down in a much simpler way. Our solutions would actually appear as follows.

(a) 32.7×0.00764
$= \underline{0.250}$

number	log
32.7	1.5145
0.00764	$\bar{3}.8831$
0.2498	$\bar{1}.3976$

(b) $9.64 \div 23.8$
$= \underline{0.405}$

number	log
9.64	0.9841
23.8	1.3766
0.4050	$\bar{1}.6075$

(c) $4.29^4 = \underline{339}$

number	log
4.29	0.6325
×4	2.5298

Note how we write logarithms. They consist of a whole number part , called the CHARACTERISTIC which corresponds to the power of 10 in the standard form for the number, (and is written with a bar over the top if negative) and a decimal part, the MANTISSA which is the power of 10 corresponding to the number between 1 and 10. You must take special care when dealing with negative characteristics.

(d) $\sqrt[3]{172} = \underline{5.56}$

number	log
172	2.2355
÷3	0.7452

These examples also illustrate the laws of logarithms which are really the same as the laws of indices
(1) $\log(a \times b) = \log a + \log b$ (2) $\log(a \div b) = \log a - \log b$ (3) $\log\left(a^b\right) = b \times \log a$

We do not need a law for obtaining roots since division is just the same as multiplying by a fraction.

Ex. (ii) Solve the equation $5^x = 4$

This is an example of one of the important uses of logarithms even in the current calculator age as it can only be done by taking logarithms of each side to give $x \log 5 = \log 4$ using law (3) above.

Hence, $x = \frac{\log 4}{\log 5} = \frac{0.6021}{0.6990} = \underline{0.861}$

Be careful not to confuse $\log\left(\frac{a}{b}\right)$ with $\frac{\log a}{\log b}$

Ex. (iii) Express $\log\left(\frac{x^3}{\sqrt{y}}\right)$ in terms of $\log x$ and $\log y$

Using the laws of logarithms we have $\log\left(\frac{x^3}{\sqrt{y}}\right) = \log\left(x^3\right) - \log\left(\sqrt{y}\right) = \underline{3\log x - \frac{1}{2}\log y}$

Logarithms to other bases are usually written with the base as a subscript, thus $\log_2 8 = 3$ is the statement that the logarithm of 8 to base 2, (i.e. the power of 2 that is equal to 8), is 3

Note that the statements $\log_a x = n$ and $a^n = x$ are equivalent to each other and you should make sure that you can easily change from one form to the other.

Ex. (iv) Find the logarithms to base 4 of (a) 16 (b) 2 (c) 1 (d) $\frac{1}{4}$ (e) 4 (f) 8

(a) $16 = 4^2 \Rightarrow \underline{\log_4 16 = 2}$ (b) $2 = \sqrt{4} = 2^{\frac{1}{2}} \Rightarrow \underline{\log_4 2 = \frac{1}{2}}$ (c) $1 = 4^0 \Rightarrow \underline{\log_4 1 = 0}$

(d) $\frac{1}{4} = 4^{-1} \Rightarrow \log\frac{1}{4} = -1$ or $\log\frac{1}{4} = \log 1 - \log 4 = 0 - 1 = -1$ (since $\log_a 1 = 0$ for all bases)

(e) $4 = 4^1 \Rightarrow \underline{\log_4 4 = 1}$ (Note that the logarithm of the base is always 1)

(f) $8 = 2^3 = (\sqrt{4})^3 = \left(4^{\frac{1}{2}}\right)^3 = 4^{\frac{3}{2}} \Rightarrow \underline{\log_4 8 = \frac{3}{2}}$

Ex. (v) Express $\frac{1}{3}\log 8 - \log\left(\frac{2}{5}\right)$ as a single logarithm.

$\frac{1}{3}\log 8 - \log\left(\frac{2}{5}\right) = \log\left(8^{\frac{1}{3}} \div \frac{2}{5}\right) = \log\left(2 \div \frac{2}{5}\right) = \underline{\log 5}$

Ex. (vi) Solve the equation $\log_6 x + \log_6(x+5) = 2$

$\log_6 x + \log_6(x+5) = 2 \Rightarrow \log_6[x(x+5)] = \log_6 36 \Rightarrow x(x+5) = 36$

i.e $x^2 + 5x - 36 = 0 \Rightarrow (x+9)(x-4) = 0 \Rightarrow x = -9$ or 4

However, $\log_6 x$ and $\log_6(x+5)$ are undefined for $x = -9$ and so the only valid solution is $\underline{x = 4}$

Logarithmic - Index Form

Since $\log y = x \Rightarrow y = 10^x$ we vcan see how to convert from one form to the other. In general if $y = a^x$ then $\log_a y = x$ i.e. the logarithm to base a is equal to x. This applies for logarithms to any base.

Solving equations using logarithms.

When the unknown quantity in an equation is in a power we usually have to use logarithms to solve it.

Ex. Find the value of x when $2^{2x+1} - 5(2^x) + 2 = 0$

This may be rewritten as $2\left(2^{2x}\right) - 5(2^x) + 2 = 0 \Rightarrow (2(2^x) - 1)(2^x - 2) = 0$ note that $2^{2x} = (2^x)^2$

so $2^x = \frac{1}{2}$ or 2

We now take logarithms so $x\log 2 = \log\frac{1}{2}$ or $x\log 2 = \log 2 \Rightarrow x = \frac{\log(1/2)}{\log 2}$ or 1

Note also that $\log\left(\frac{1}{2}\right) = \log 1 - \log 2 = -\log 2$ since $\log 1 = 0$ so $x = \pm 1$

When using logarithms however, never forget that the logarithm of a number smaller than 1 is negative.

Reduction of an exponential equation to a linear form

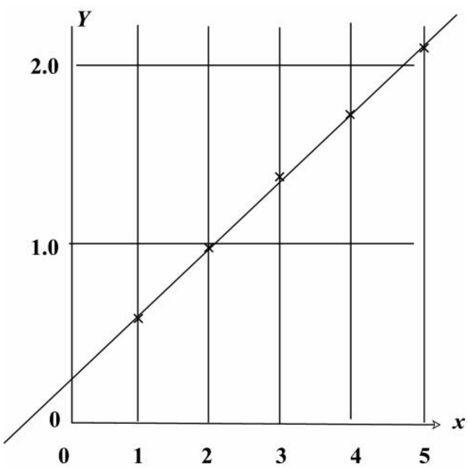
A LINEAR RELATION is a relation of the form $y = \mathrm{f}(x)$ such that the graph of y against x is a straight line. i.e. $\mathrm{f}(x)$ must have the form $ax + b$. Many nonlinear relations can be transformed into linear ones by suitable substitutions. Drawing a graph can then enable us to find the values of unknown constants.

Ex. Consider the relation $y = ax^n$, by replacing x^n by X we have $y = aX$ which has a straight line graph, the gradient of which will be the value of a.

Similarly, $y = ab^x$ is reduced to $Y = Ax + B$ by taking logarithms to give $\lg y = \lg a + x\lg b$ and then putting $Y = \lg y$ and plotting Y against x. The gradient of the graph will be the value of $\lg b$ whilst $\lg a$ will be the intercept on the y -axis. Note! You can take ordinary logs or natural logs, it doesn't really matter.

Ex. It is thought that x and y are related by an equation of the form $y = ab^x$ Given the following set of values for x and y, estimate the values of a and b by drawing a suitable linear graph
We first take logs to produce $\log y = \log a + x \log b$ and
Plot values of $\log y$ against x

x	1	2	3	4	5
y	3.8	9.2	22.1	53.1	127.4
Y=lg$y$	0.58	0.96	1.34	1.73	2.11



We can see from the graph that the Intercept on the y-axis is at 0.2 so $\log a \simeq 0.2 \Rightarrow a \simeq 1.6$
Using the points $(1, 0.58)$ and $(5, 2.11)$ the gradient is
$\frac{2.11-0.58}{4} = 0.38$ and so $\log b \simeq 0.38 \Rightarrow b \simeq 2.4$
ence, x and y obey a law of the form

$\underline{y = 1.6(2.4)^x}$

Sequences and Series

A function whose domain is the set of natural numbers is called a SEQUENCE. The elements of the range of the function are the TERMS of the sequence. There are two common methods of defining a sequence

(1) By giving a formula for the general term. Which again is done in one of two ways.

Ex. $f(n) = 2n - 1$ defines the sequence 1, 3, 5, 7, ...

Ex. $\{n^3 - 1\}$ defines the sequence 0, 7, 26, 63, ...

(2) By expressing the n^{th} or $(n + 1)^{th}$ term in terms of the previous term or sometimes the previous two Terms. u_n is commonly used to denote the n^{th} term of a sequence.

Ex. $u_{n+1} = nu_n$ with $u_1 = 1$ defines the sequence 1, 1, 2, 6, 24, ...

Ex. $u_{n+1} = 3u_n - u_{n-1}$ defines the sequence 1, 1, 2, 5, 13, ...

These are called RECURRENCE RELATIONS

Series

The sum of the first n terms of a sequence forms a FINITE SERIES. If all the terms of a sequence are added we have an INFINITE SERIES

e.g. $1 + 2 + 4 + 8 + 16 + 32$ is a finite series of six terms.

$1 + 2 + 4 + 8 + 16 + 32 + ... + 2^n + ...$ is an INFINITE SERIES

Behaviour of series

Sequences and series may behave in a number of ways as the number of terms increase. The sum of the series may

(i) Tend to some finite limiting value. We say that it CONVERGES to that value

(ii) Increase in size without limit. We say it DIVERGES

(iii) OSCILLATE between finite of infinite limits.

(iv) Display PERIODICITY. i.e the same sequence of values repeat at regular intervals.

Ex. (i) $1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{n+1}} + ...$ converges to the value 2. We say the sum to infinity is 2

(ii) $1 + 2 + 4 + 8 + 16 + ... + 2^n + ...$ diverges. The sum tends to infinity.

(iii) $11 - -2 + 4 - 8 + 16 - ...$ oscillates

(iv) $\{1 - (-1)^n\} = \{2, 0, 2, 0, ...\}$ displays periodicity

Note that a sequence is a collection of numbers whereas a finite series has a unique finite numerical value. An infinite series may or may not have a finite value.

We may occasionally relax the requirement that the domain of our function be the set of natural numbers and allow integer value or omit the first few values. We commonly use the sigma notation when dealing with series. i.e. $\sum_{r=1}^{n} f(r)$ which means that we add together the values of f(r) for all integer values of r between 1 and n inclusive.

Ex. (i) $\sum_{r=1}^{r=4} r = 1 + 2 + 3 + 4 = 10$ (ii) $\sum_{1}^{5}(3r - 1) = 2 + 5 + 8 + 11 + 14 = 40$

Note also that we often omit the $r =$ when it can cause no confusion.

(iii) $\sum_{3}^{6}(r^2 - 5) = 4 + 111 + 20 + 31 = 66$

Note also that the same series may be defined in many different ways

Ex. $\sum_{1}^{n} r(r - 1) = \sum_{0}^{n-1} r(r + 1) = \sum_{2}^{n+1} (r - 1)(r - 2)$ etc.

Arithmetic sequences and series (APs)

If $u_r - u_{r-1} = d$ (a constant), then we say that $\{u_r\}$ is an ARITHMETIC SEQUENCE or PROGRESSION (AP). D is the COMMON DIFFERENCE and it is usual to denote the first term by a. Thus the general AP takes the form $a, a+d, a+2d, \ldots$

By inspection it should be fairly obvious that the general or n^{th} term is given by $u_n = a + (n-1)d$

We use S_n to denote the sum of the first n terms, hence $S_n = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d)$

Clearly also, by writing the terms in reverse order, $S_n = \{a+(n-1)d\} + \{a+(n-2)d\} + \ldots + (a+d) + a$

Adding term by term we have $2S_n = [2a+(n-1)d] + [2a+(n-1)d] + \ldots$ to n terms

Thus $S_n = \frac{1}{2}n[2a+(n-1)d]$ or $\frac{1}{2}n(u_1 + u_n)$

In particular $\sum_1^n r = 1+2+3+4+\ldots+n = \frac{1}{2}n(n+1)$ as stated in the previous exercise.

Ex. The eighth term of an AP is – 21 and the fifteenth term is – 49. Calculate the sum of the first twenty terms.

We have $u_8 = a + 7d = -21$ and $u_{15} = a + 14d = -49$ hence $7d = -28 \Rightarrow d = -4$ so $a = 7$

We then have $S_{20} = 10\{14 + 19(-4)\} = 10 \times (-62) = -620$

Geometric sequence and series

If $u_n = ru_{n-1}$ (r constant) then we say that $\{u_n\}$ is a GEOMETRIC SEQUENCE or GEOMETRIC PROGRESSION. (GP) r is the COMMON or CONSTANT RATIO, and the sequence has the general form $a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$

The general term $u_n = ar^{n-1}$

Clearly $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \Rightarrow rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$

Subtracting one form from the other we have $(1-r)S_n = a - ar^n$ all other terms cancelling, hence

$S_n = \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$ (providing $r \neq 1$) the former being used when $|r| < 1$, the latter when $|r| > 1$.

Clearly, if $r = 1$ then $S_n = na$

For the case when $|r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$ and so $S_n \to \frac{a}{1-r}$ This result, known as the sum to infinity, is especially important.

Ex.(i) Which term of $\{2, 4, 8, ..\}$ is equal to 512 ?

We have $a = 2,\ r = 2$ and $u_n = 512 \Rightarrow 2 \times 2^{n-1} = 512 \Rightarrow 2^n = 512 \Rightarrow n = 9$

So 512 is the 9th term of the sequence.

Ex (ii) Find (a) the sum of the first 5 terms (b) the sum to infinity, of the geometric series $9 - 3 + 1 - \ldots$

(a) We have $a = 9,\ r = -\frac{1}{3} \Rightarrow S_5 = \frac{9[1-(-\frac{1}{3})^5]}{1-(-\frac{1}{3})} = \frac{3}{4} \times 9[1 + \frac{1}{243}] = \frac{3}{4} \times 9 \times \frac{244}{243} = \frac{61}{9} = 6\frac{7}{9}$

(b) It is usual to simply use for the sum to infinity, so $S = \frac{9}{1-(-\frac{1}{3})} = \frac{3}{4} \times 9 = 6\frac{3}{4}$

Ex (iii) How many terms of the geometric series $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$ must be taken for the sum to exceed 5.99 ?

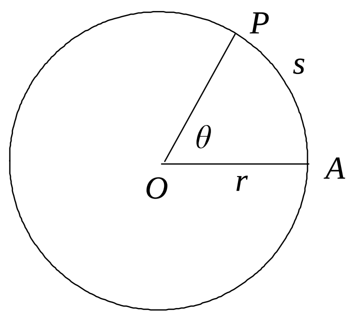
We require $S_n = \frac{2[1-(\frac{2}{3})^n]}{1-\frac{2}{3}} > 5.99 \Rightarrow 6[1 - (\frac{2}{3})^n] > 5.99 \Rightarrow 6 - 5.99 > 6\left(\frac{2}{3}\right)^n$

i.e. $\left(\frac{2}{3}\right)^n < \frac{0.01}{6} = 0.001667$ The only way to solve an equation such as this in which the unknown appears as an index, other than laborious trial and error, is by using logarithms. So, taking logarithms of both sides we have $n\log\left(\frac{2}{3}\right) < \log 0.001667 \Rightarrow -0.176n < -2.778$ Note, we have here taken logarithms to base 10, the so called common logarithms, denoted by log on the calculator, but you can equally well use logarithms to base e denoted ln on your calculator.

Hence, $n > \frac{2.778}{0.176} = 15.8$ so we must take at least 16 terms of the sequence. Remember that we must change the direction of the inequality when dividing by a negative number. We could have avoided this problem by inverting $\left(\frac{2}{3}\right)^n < \frac{0.01}{6}$ to give $\left(\frac{3}{2}\right)^n > \frac{6}{0.01} = 600,$ so $n \log 1.5 > \log 600$.

Trigonometry

Previously, in your GCSE course, you will have worked mainly in degrees and decimals of a degree (or possibly minutes), for the measurement of angles, but at A-level, we more often have to use an alternative measure of angle, the RADIAN.



Suppose an arm OP starts from the position OA and rotates in an anti-clockwise direction through an angle denoted by θ

The measure of θ in radians is defined to be the ratio of the length of the arc AP to the radius OA . ie $\theta = \frac{s}{r}$ radians

A radian is therefore the angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

Note that the radian is a dimensionless measure, a pure number.

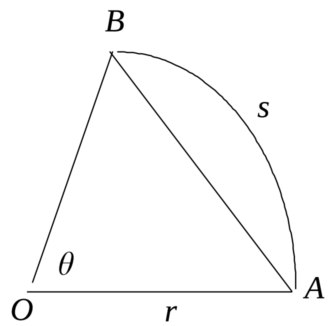
Clearly, $360^{\mathrm{o}} = \frac{2\pi r}{r} = 2\pi$ radians $\Rightarrow$ 1 radian $= \frac{180^{\mathrm{o}}}{\pi} = 57.3^{\mathrm{o}}$

It is essential to be able to convert quickly from degrees to radians and vice versa.

Ex. $50^{\mathrm{o}} = 50 \times \frac{\pi}{180} = \frac{5\pi}{18}$ radians whilst $\frac{7\pi}{4}$ radians $= \frac{7\pi}{4} \times \frac{180}{\pi} = 315^{\mathrm{o}}$

Try to memorise the radian equivalents of the more common angles such as $30^{\mathrm{o}}, 45^{\mathrm{o}}, 60^{\mathrm{o}}, 90^{\mathrm{o}}$ and multiples of these.

A big advantage of radian measure is that the expressions for length of arc and areas of sectors and segments of circles take much simpler forms than when working in degrees.



Remember that the area of a triangle of sides a and b at an angle of θ is given by $\frac{1}{2}ab\sin\theta$

If $A\hat{O}P = \theta$ radians then arc $AB = s = r\theta$

Area of sector $AOB = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}r^2\theta$

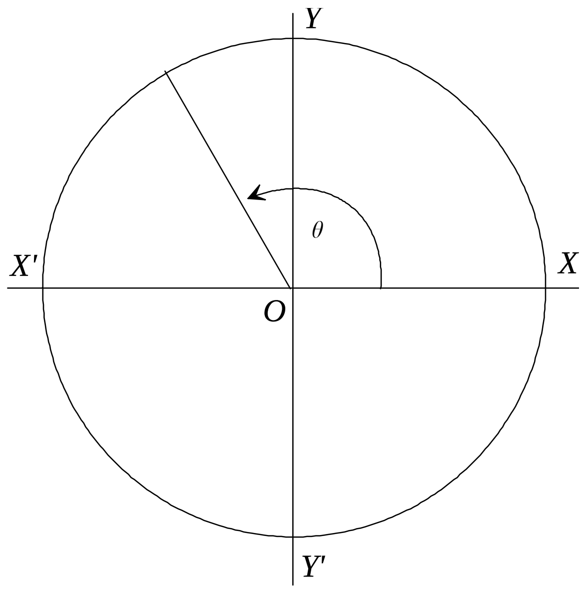
Area of segment = area of sector – area of $\Delta AOB = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

The corresponding formulae when θ is measured in degrees are

$s = \frac{\pi r\theta}{180}$ area of sector $= \frac{\pi r^2\theta}{180}$ area of segment $= \frac{1}{2}r^2\left(\frac{\pi\theta}{90} - \sin\theta\right)$

The Trigonometric Functions



Let $X'OX$, $Y'OY$ be perpendicular axes in the usual way. Consider a line of unit length, initially in the position OX, and let this line rotate about O through an angle θ to arrive at OP. Note that it is conventional to take an anti-clockwise rotation to be positive and a clockwise rotation to represent negative angles.

Let the coordinates of P be (x, y) then, in both sign and magnitude, we define the sine, cosine and tangent of any angle by

$\cos\theta = x$, $\sin\theta = y$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$ $(x \neq 0)$

More generally, of course, if OP is not of unit length then we could define $\cos\theta = \frac{x}{OP}$ etc.

Clearly, since a rotation through any integral multiple of 360º (or 2π radians) will bring OP back to the same position, we have $\sin(\theta \pm 360k)^{o} = \sin\theta^{o}$ or $\sin(\theta \pm 2k\pi) = \sin\theta$ for any integral k.

Special Angles.

Angles such as 30°,45°,60°,90° and their multiples occur very frequently in examination questions and have values of sin, cos and tan that can be expressed in terms of simple surds. You should memorize these values. Draw an equilateral triangle of side 2 units and an isosceles right-angled triangle with the equal sides of unit length, Deduce the sine, cosine and tangent of the above angles, giving your answers in surd form with rational denominators.

To find trigonometric functions of any angle

We do this in two simple steps.

(1) Find whether the result is to be positive or negative by considering the position of the point P corresponding to the given angle. The diagram on the right shows which functions are positive in each quadrant.

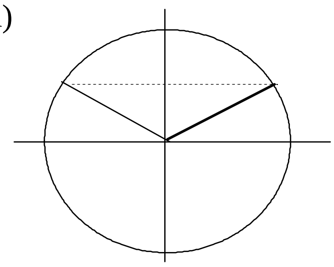
Sin	All
Tan	Cos

(2) Find the acute angle which has the same numerical values of x and y

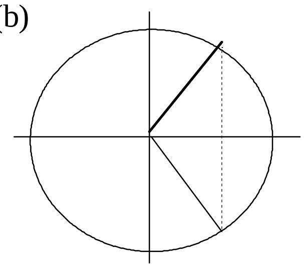
This is always the acute angle between OP and the x-axis.

Ex. Find (a) sin150° (b) tan310° (c) cos(– 115°)

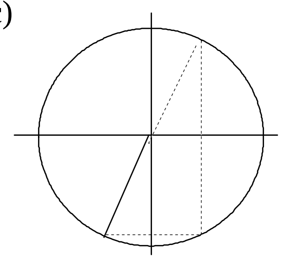
(a)



(b)



(c)



$\sin 150^{o} = \sin 30^{o} = 0.5$ $\quad \tan 310^{o} = -\tan 50^{o} = -1.192$ $\quad \cos(-115^{o}) = -\cos 65^{o} = -0.423$

To find all angles within a specified range having a given value of a trigonometric function

There are four steps involved this time.

(1) Find the acute angle corresponding to the modulus of the function

(2) Determine the quadrants in which the angle may lie

(3) Modify the given range to suit the angle involved in the function

(4) Find all angles in the modified range and answer the question.

Ex. Find all angles in the range 0° to 360° such that (a) $\cos x = 0.4$ (b) $\tan 2x = -1.2$ (c) $\sin\left(\frac{1}{2}x\right) = 0.6$

(a) Step 1. $\cos^{-1}0.4 = 66.4^{o}$ (Note, $\cos^{-1}$ means ‘the angle whose cosine is’)

Step 2. Cosine is positive in the first and fourth quadrants

Step 3. No range modification required.

Step 4. $x = 66.4^{o}$ or $360 - 66.4 = 293.6^{o}$

(b) $\arctan 1.2 = 50.2^{o}$ (Note, arctan is an alternative notation for $\tan^{-1}$)

Tangent is negative in the second and fourth quadrants

$0^{o} < x < 360^{o} \Rightarrow 0^{o} < 2x < 720^{o}$

$2x = 180 - 50.2 = 129.8^{o}$ or $360 - 50.2 = 309.8^{o}$ or $540 - 50.2 = 489.8^{o}$ or $720 - 50.2 = 669.8^{o}$

Hence, $x = 64.9^{o}, 154.9^{o}, 244.9^{o}$ or 334.9^{o}

(c) This is how we would normally set down a solution:

$0^{o} < x < 360^{o} \Rightarrow 0^{o} < \frac{1}{2}x < 180^{o}$ and $\sin^{-1}0.6 = 36.87^{o}$

sine is positive in the first and second quadrants

Hence, $\frac{1}{2}x = 36.87^o$ or $180 - 36.87 = 143.13^o \Rightarrow x = 73.7^o$ or 286.3^o

Note that it was necessary to find $\frac{1}{2}x$ correct to 2 dp to ensure that x is correct to 1 dp.

The first fundamental trigonometric identity

Applying Pythagoras's theorem to a right angled triangle with sides a, b and c where c is the hypotenuse we obviously have $a^2 + b^2 = c^2$ so dividing by c^2 this gives $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$ i.e. $(\sin\theta)^2 + (\cos\theta)^2 = 1$ where θ is the angle between a and b. Note that $(\sin\theta)^2$ is more usually written as $\sin^2\theta$

This ientity enables us to solve a certain type of equation.

Ex. Solve the equation $3\sin^2 x + 2\cos x = 2$ for values of x from 0^o to 360^o

We use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain a quadratic equation in $\cos x$

i.e. $3 - 3\cos^2 x + 2\cos x = 2 \Rightarrow 3\cos^2 x - 2\cos x - 1 = 0 \Rightarrow (3\cos x + 1)(\cos x - 1) = 0 \Rightarrow \cos x = -\frac{1}{3}$ or 1

This is then completed as described earlier to give $x = 0^o, 109.5^o, 250.5^o$ or 360^o

The Sine and Cosine Rules

You should be familiar with these and their use in solving non right angled triangle from you GCSE Maths course. Here we will just recap them.

Sine Rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine Rule $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \frac{b^2+c^2-a^2}{2bc}$

Calculus

The CALCULUS (from the Latin for pebble) was developed to deal with two fundamental problems:

(i) finding the slope, or gradient of a curve

(ii) finding the area 'under' a curve, ie between a curve and the axis.

It has since proved to be one of the most useful branches of mathematics with applications in engineering, all sciences, economics and many other fields.

Let us begin by considering a simple problem of the first type.

Fitting a curve to experimental data suggests that the distance fallen from rest by a small heavy object is related to the time, to a fair degree of accuracy, by the equation $s = 5t^2$ where s metres is the distance fallen in time t seconds. How is the speed of the object related to the time ?

You should have met distance - time graphs before and you may remember that the speed at any given time is measured by the gradient of the tangent to the graph at that point. Since the slope of a curve may be defined to be the gradient of the tangent to the curve we clearly have a problem of type (i).

Consider first the speed at a given instant of time, say after 2 seconds.

From $t = 2$ to $t = 3$ the distance fallen is $5(3)^2 - 5(2)^2 = 25$ m. Hence, in this 1 second interval, the 'average' speed is $25\ \text{ms}^{-1}$. This will not be a very good estimate of the speed at $t = 2$ however since the speed will change appreciably during this 1 second period. To get a better estimate we consider a shorter interval of time starting at $t = 2$. Say from $t = 2$ to $t = 2.1$, giving a distance fallen of $5(2.1)^2 - 5(2)^2$ ie 2.05 m, and hence an average speed of $\frac{2.05}{0.1} = 20.5\ \text{ms}^{-1}$. Taking shorter and shorter intervals we can obtain more and more accurate estimates of the actual speed at $t = 2$.

We now generalise the above process thus:

In the time interval from $t = 2$ to $t = 2 + h$, where h is an arbitrarily small number, the distance fallen is given by $5(2 + h)^2 - 5(2)^2 = 20 + 20h + 5h^2 - 20 = 20h + 5h^2$ and so the average speed during this interval is $\frac{20h+5h^2}{h} = 20 + 5h\ \text{ms}^{-1}$

It should now be clear that by taking smaller and smaller values of h, the average speed will get closer and closer to 20 ms^{-1} and can be made as close to 20 ms^{-1} as we please by taking a sufficiently small value of h. We therefore say that the speed is 20 ms^{-1} when $t = 2$.

The next stage of generalisation is to repeat the process for an arbitrary value of the time. Taking t to represent the time we use δt (read as 'delta-t') to represent a small increment in the value of t. (Generally, a δ put in front of a variable indicates a small increment in the value of that variable).

Then in the time interval from t to $t + \delta t$ the distance fallen is $5(t + \delta t)^2 - 5t^2 = 10t.\delta t + 5(\delta t)^2$

And the average speed is thus $\frac{10t.\delta t+5(\delta t)^2}{\delta t} = 10t + 5\delta t$ The speed at time t is thus $10t$.

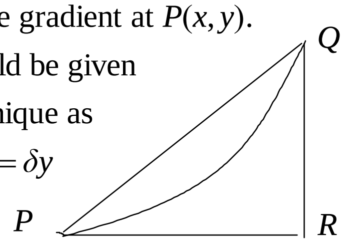
Note that the square of δt must be written with brackets thus $(\delta t)^2$ and not δt^2 since this would mean 'a small increment in the value of t^2'

Differentiation

We now consider the more general problem of finding the gradient of a curve.

Consider a curve with equation $y = f(x)$ and we wish to find the gradient at $P(x, y)$.

If this were a distance time graph then the average speed would be given by the slope of the chord PQ and so, following the same technique as before we have, if Q is the point $(x + \delta x, y + \delta y)$, $PR = \delta x$, $RQ = \delta y$ and so gradient of $PQ = \frac{\delta y}{\delta x}$



So the gradient of the curve at $P = \lim_{Q\to P}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)$ This statement is read as 'the limiting value of delta-y by delta-x as delta-x tends to zero'. Providing this limit exists, we denote it by the symbol $\frac{\mathrm{d}y}{\mathrm{d}x}$ read as 'd-y by d-x' and is called the DERIVATIVE or DIFFERENTIAL COEFFICIENT of y with respect to x.

ie. $\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$ This really defines the derivative to be the rate of change of y with respect to x.

Note that $\frac{dy}{dx}$ is NOT a fraction. It is the limiting value of a fraction.

Ex. Find the gradient of $y = 2x^2 - 3x + 2$ at the point (2, 5)

If $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ are neighbouring points on the curve then we have

$y = 2x^2 - 3x + 2$ and $y + \delta y = 2(x + \delta x)^2 - 3(x + \delta x) + 2$ so by subtraction

$\delta y = 4x\delta x + 2(\delta x)^2 - 3\delta x$ and $\frac{\delta y}{\delta x} = 4x + 2\delta x - 3$

Thus $\frac{dy}{dx} = \lim_{\delta x \to 0}(4x + 2\delta x - 3) = 4x - 3$ and so the gradient at (2, 5) is 8 – 3 = 5

Equations of Tangents and Normals

Now that we can find the gradient of a curve at a given point it is easy to obtain the equations of the tangent and normal to the curve at that point.

Ex Find the equations of the tangent and normal to the curve $y = 3x^2 - 4x + 5$ at the point $(2, 9)$

We first differentiate to obtain the gradient function $\frac{dy}{dx} = 6x - 4$ so gradient at $(2, 9)$ is 8

Using $y - k = m(x - h)$ equation of tangent is $y - 9 = 8(x - 2)$ i.e. $y = 8x - 7$

Since the normal is perpendicular to the tangent it's gradient is $-\frac{1}{8}$ so equatuion of normal is

$y - 9 = -\frac{1}{8}(x - 2) \Rightarrow 8y - 72 = -x + 2$ or $x + 8y = 74$

Alternative notation

The 'function' notation is often used as an alternative to δy and δx.

We use the notation $f: x \to 3x^2 + 2x$ to define a function f which assigns to each value of x, the value of $3x^2 + 2x$. We then write the image of x under this mapping as $f(x)$

So we write $f(x) = 3x^2 + 2x$ and then $f(2) = 3(2)^2 + 2 \times 2 = 16$ and $f(-3) = 3(-3)^2 + 2(-3) = 21$ etc.

With this notation, if a and $a + h$ are neighbouring values of x, then $f(a + h) - f(a)$ is the change in the value of the function corresponding to the change h in x.

Hence, gradient of $f(x)$ at $x = a$ is given by $\lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$ and we denote the gradient function by $f'(x)$ which we read as 'f-dash of x' and so in general $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x))}{h}\right)$

Ex. Find the gradient function of $f(x) = \frac{1}{x+1}$ $(x \neq -1)$

Using the above result, $f'(x) = \lim_{h \to 0}\left(\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h}\right) = \lim_{h \to 0}\left(\frac{(x+1)-(x+h+1)}{h(x+1)(x+h+1)}\right) = \lim_{h \to 0}\left(\frac{-1}{(x+1)(x+h+1))}\right) = \frac{-1}{(x+1)^2}$

The process of finding the gradient function is known as DIFFERENTIATION and the method we have used so far is called differentiation from first principles. Obviously we do not want to have to find all such functions as laboriously as this. It is time we developed a few rules to help us.

(1) Derivative of x^n for integral values of n. During the preceding exercises you should have discovered the following results

$f(x) = x^2 \Rightarrow f'(x) = 2x$: $f(x) = x^3 \Rightarrow f'(x) = 3x^2$: $f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2}$: $f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3}$

These results suggest the rule $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$. We will assume for the time being that this result holds for all values of n. In words 'to differentiate a power of x, multiply by the power and decrease the power by one' thus $f(x) = x^{2/3} \Rightarrow f'(x) = \frac{2}{3}x^{-1/3}$ and $f(x) = x^{-6} \Rightarrow f'(x) = -6x^{-7}$ etc.

(2) Derivative of a Constant. $f(x) = c \Rightarrow f(x + h) = c \Rightarrow f(x + h) - f(x) = 0 \Rightarrow f'(x) = 0$

(3) Derivative of Ax^n where A is a constant.

$f(x) = Ax^n \Rightarrow f(x + h) - f(x) = A(x + h)^n - Ax^n = A\{(x + h)^n - x^n\} \Rightarrow f'(x) = Anx^{n-1}$

Thus, multiplying constants are not affected by differentiation. In general, for any function $f(x)$ the derivative of $Af(x)$ is $Af'(x)$

(4) Derivative of $f(x)+g(x)$. Examination of examples done previously suggest that this is $f'(x)+g'(x)$
And it is not too difficult to prove this from first principles.

The following examples also illustrate the three acceptable methods of presentation. Correct notation is very important if you are to avoid errors later on.

Ex Find $\frac{dy}{dx}$ if $y = 4x^3 + x^2 - 5x + 6$

$y = 4x^3 + x^2 - 5x + 6 \Rightarrow \frac{dy}{dx} = 12x^2 + 2x - 5$

Ex Find the derivative of $(3x-2)^2$

Let $f(x) = (3x-2)^2 = 9x^2 - 12x + 4 \Rightarrow f'(x) = 18x - 12$

Ex Differentiate $\frac{\sqrt{x}+2}{x}$

$\frac{d}{dx}\left(\frac{\sqrt{x}+2}{x}\right) = \frac{d}{dx}\left(x^{-1/2} + 2x^{-1}\right) = -\frac{1}{2}x^{-3/2} - 2x^{-2} = -\frac{\sqrt{x}+4}{2x^2}$

Increasing and decreasing functions

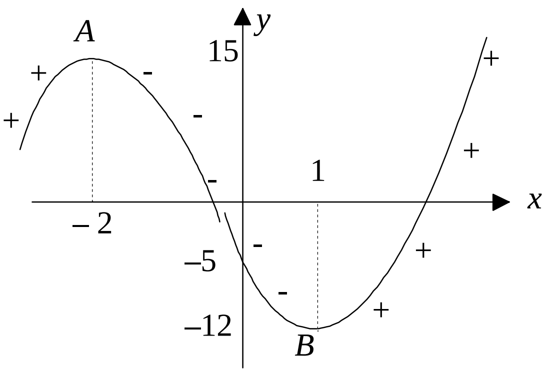
$f'(x) > 0 \Rightarrow f(x)$ is increasing whilst $f'(x) < 0 \Rightarrow f(x)$ is decreasing

Ex. $f(x) = 2x^3 + 3x^2 - 12x - 5 \Rightarrow f'x) = 6x^2 + 6x - 12 = 6(x-1)(x+2)$

Thus $x < -2$	$\Rightarrow$	$f'(x) > 0$	$\Rightarrow$	$f(x)$ is increasing
$-2 < x < 1$	$\Rightarrow$	$f'(x) < 0$	$\Rightarrow$	$f(x)$ is decreasing
$x > 1$	$\Rightarrow$	$f'(x) > 0$	$\Rightarrow$	$f(x)$ is increasing

When $x = -2$ or 1, $f'(x) = 0$ and so the tangents to the curve at these points are parallel to the x-axis.

The coordinates of these points are $(-2, 15)$ and $(1, -12)$.
Clearly also, $x = 0 \Rightarrow y = -5$. With this information it is easy to 'sketch' the graph of $f(x)$.



Note. The + and - signs refer to the sign of $f'(x)$

Stationary points

Points such as A and B in the above diagram, ie where $f'(x) = 0$, are called STATIONARY POINTS since the function is neither increasing nor decreasing at these points. A, where the function takes a value greater than that at any neighbouring point, is called a (local) MAXIMUM TURNING POINT, whilst B is a (local) MINIMUM TURNING POINT. We can easily see that the condition for a maximum point is that the derivative must change from positive to negative, whilst at a minimum point it must change from negative to positive.

Ex. (viii). Find the stationary points on the curve $y = x^3 - 4x^2 - 3x + 2$ and state whether they are maximum or minimum turning points.

We have $\frac{dy}{dx} = 3x^2 - 8x - 3 = (3x+1)(x-3)$ so $\frac{dy}{dx} = 0$ when $x = -\frac{1}{3}$ or 3

If x is just less than $-\frac{1}{3}$ (ie $-\frac{1}{3}$ and a bit) then $3x + 1 < 0$ and $x - 3 < 0 \Rightarrow \frac{dy}{dx} > 0$

If x is just greater than $-\frac{1}{3}$ then $3x + 1 > 0$ and $x - 3 < 0 \Rightarrow \frac{dy}{dx} < 0$

The inverse of differentiation

Given $f'(x)$, can we find $f(x)$? If we can find a function $f(x)$ whose derivative is $f'(x)$ then we say we have found a PRIMITIVE of $f'(x)$.

Ex. (i). $f'(x) = 10x - 3x^2$

It is reasonably obvious that $f(x) = 5x^2 - x^3$ is a possible primitive of $10x - 3x^2$, but clearly, so also are $5x^2 - x^3 + 7$ or $5x^2 - x^3 - 6$ etc. In fact , $5x^2 - x^3 + c$ is a primitive for any value of c. The primitives thus form a family of parallel curves, ie curves having the same gradient for any given value of x.

If we are given one pair of values of x and y, ie the coordinates of one point on the curve, then we can determine the particular primitive.

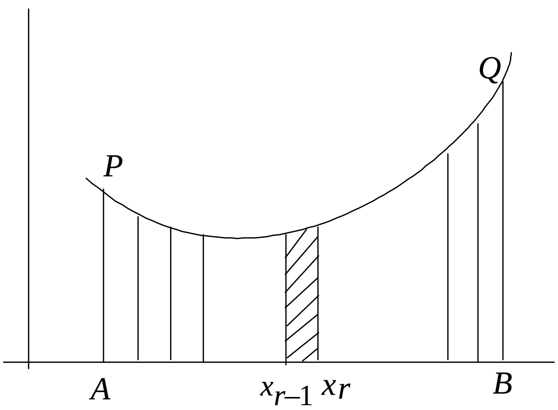
Ex. (ii). Find the equation of the curve with gradient function $x^2 - 3$ which passes through the point (2, 5)

$\frac{dy}{dx} = x^2 - 3 \Rightarrow y = \frac{1}{3}x^3 - 3x + c$ and $y = 5$ when $x = 2 \Rightarrow 5 = \frac{8}{3} - 6 + c \Rightarrow c = \frac{25}{3}$

So the equation of the curve is $y = \frac{1}{3}x^3 - 3x + \frac{25}{3}$ or $3y = x^3 - 9x + 25$

The area 'under' a curve

We now come to the second basic problem for which calculus has been developed. Finding the area between a curve $y = f(x)$, the x - axis and ordinates $x = a, x = b$



Let AB be divided into n parts by the points with x coordinates $x_0(=a), x_1, x_2, \ldots x_{n-1}, x_n(=b)$

An approximation to the required area is then given by $\sum_{r=1}^{n}(x_r - x_{r-1})f(\theta_r)$ where $x_{r-1} < \theta_r < x_r$

If we now let $n \to \infty$ in such a way that each $x_r - x_{r-1} \to 0$ we have

Area = $\lim_{n\to\infty} \sum_{r=1}^{n}(x_r - x_{r-1})f(\theta_r)$

Providing this limit exists it is denoted by $\int_a^b f(x)dx$ and is called the INTEGRAL of $f(x)$ with respect to x from $x = a$ to $x = b$.

But how do we evaluate an integral.? Consider an alternative approach to finding the area. Let A denote the area up to $x = x_{r-1}$ let $x_r = x$, $x_r - x_{r-1} = \delta x$ and let δA denote the area shown shaded. It should be clear that $f(x)\delta x \leqslant \delta A \leqslant f(x+\delta x)\delta x$ so dividing through by δx we have

$f(x) \leqslant \frac{\delta A}{\delta x} \leqslant f(x+\delta x)$ Now let $\delta x \to 0$ then $x + \delta x \to x$ and $\frac{\delta A}{\delta x} \to \frac{dA}{dx} \Rightarrow \frac{dA}{dx} = f(x)$

If the function is decreasing, rather than increasing as shown, then the inequality signs are reversed but the result is unchanged. We thus deduce that $A = F(x) + c$, a primitive of $f(x)$. Since we must have $A = 0$ when $x = a$ it follows that $c = F(a)$ and so $A = F(x) - F(a)$ and the required area is $F(b) - F(a)$.

So to evaluate the area under a curve we have only to find a primitive of the curve function. This primitive is usually written as $\int f)(x)dx$ and is called an INDEFINITE INTEGRAL. Note that you should ALWAYS include the arbitrary constant in your answer for an indefinite integral.

We see therefore that finding an integral is simply the reverse process of differentiation and by reversing the differentiation process you should be able to see that in general $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ providing $n \neq -1$. In this case, from a previous piece of work $\int x^{-1}dx = \ln x + c$ or $\ln Ax$.

Ex. (iii). (a) $\int x^2 dx = \frac{1}{3}x^3 + c$ (b) $\int(x + \sqrt{x})dx = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + c$ (c) $\int \frac{t^2 - t}{t^2}dt = \int(1 - t^{-1})dt = t - \ln t + c$

Definite Integrals

$\int_a^b f(x)\mathrm{d}x$ is called a DEFINITE INTEGRAL, the usual notation we use is as follows

$\int_a^b f(x)\mathrm{d}x = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$ where $\mathrm{F}(x)$ is the primitive of $\mathrm{f}(x)$. Note that it is usual to omit the arbitrary constant in this case since it is cancelled out in the evaluation of $\mathrm{F}(b) - \mathrm{F}(a)$.

<u>Ex. (iv)</u>. (i) $\int_1^3 x^3\mathrm{d}x = \left[\frac{1}{4}x^4\right]_1^3 = \frac{1}{4}3^4 - \frac{1}{4}1^4 = \frac{81}{4} - \frac{1}{4} = 20$

(ii) $\int_0^2 (t^2 - 3t + 1)\mathrm{d}t = \left[\frac{1}{3}t^3 - \frac{3}{2}t^2 + t\right]_0^2 = \left(\frac{8}{3} - 6 + 2\right) - (0) = -1\frac{1}{3}$

It is important to realise that whilst the indefinite integral is a function of a variable, the definite integral is simply a numerical value depending only on the form of the function being integrated and the limits of integration. The particular symbol used for the variable within the integral is quite arbitrary.

ie $\int_1^2 x^2\mathrm{d}x = \int_1^2 t^2\mathrm{d}t = \int_1^2 \theta^2\mathrm{d}\theta$ etc.

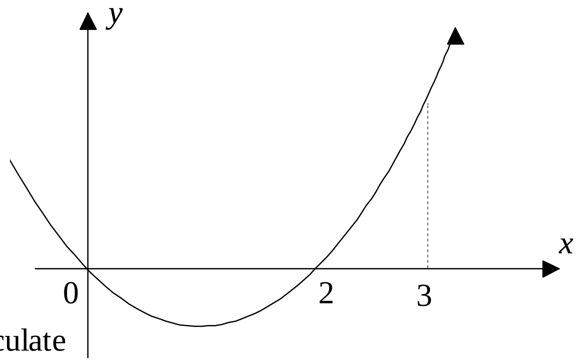
Does the value of the definite integral always give the area under the curve ?

Consider the area under $y = x^2 - 2x$ from $x = 0$ to $x = 3$

If we simply evaluate $\int_0^3 (x^2 - 2x)\mathrm{d}x$ we get $\left[\frac{1}{3}x^3 - x^2\right]_0^3 = (9 - 9) - 0 = 0$

Something is clearly wrong here. Let us look at the graph of this function.

We can immediately see that part of the area is below the axis and part above.



$\int_0^2 (x^2 - 2x)\mathrm{d}x = \left[\frac{1}{3}x^3 - x^2\right]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$

Whilst $\int_2^3 (x^2 - 2x)\mathrm{d}x = \left[\frac{1}{3}x^3 - x^2\right]_2^3 = 0 - \left(\frac{8}{3} - 4\right) = \frac{4}{3}$

So we see that the two areas are equal in magnitude but that below the axis is negative so if we require the actual area between the curve and the axis we must calculate

$\left|\int_0^2 (x^2 - 2x)\mathrm{d}x\right| + \left|\int_2^3 (x^2 - 2x)\mathrm{d}x\right| = \frac{4}{3} + \frac{4}{3} = 2\frac{2}{3}$

In any calculation of area therefore we should check whether the curve crosses the axis between the limits of the integration.

<u>Numerical Integration</u>

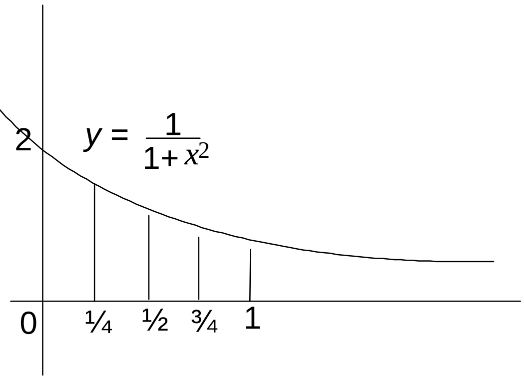
There are many times when we require the area under the curve but are unable to find the integral by analytical means. We can however, evaluate such integrals to any required degree of accuracy by numerical methods. One of the simplest to use is the TRAPEZIUM RULE. We illustrate it with an example.

<u>Ex. (v)</u> Find an approximate value for $\int_0^1 \frac{\mathrm{d}x}{1+x^2}$

Imagine the area under the curve split into

4 strips of equal width $\frac{1}{4}$ unit Each strip may be

approximated by a trapezium of width $\frac{1}{4}$ with parallel sides given by the values of $\frac{1}{1+x^2}$ at each end of the interval. Since the area of a trapezium is give by the average of the parallel sides times the distance between them, we have



$$\text{iarea} \simeq \frac{1}{2}\left(1+\frac{1}{17/16}\right)\times\frac{1}{4}+\frac{1}{2}\left(\frac{1}{17/16}+\frac{1}{5/4}\right)\times\frac{1}{4}+\frac{1}{2}\left(\frac{1}{5/4}+\frac{1}{25/16}\right)\times\frac{1}{4}$$

$$=\frac{1}{8}[1\frac{16}{17}+\frac{16}{17}+\frac{4}{5}+\frac{4}{5}+\frac{16}{25}+\frac{16}{25}+\frac{1}{2}]=\underline{0.783}$$

Later in the course you will learn how to perform this integration analytically and you will that the accurate answer is 0.7854 to 4 d.p.

<u>General result</u>

If the area under $y = \mathrm{f}(x)$ is approximated by n trapezia of equal width h then if $y_0,\ y_1,\ y_2,\ ....\ y_n$ are the ordinates at the points of division of the base of the area, we have

$$\text{area} \simeq \frac{h}{2}(y_0+y_1)+\frac{h}{2}(y_1+y_2)+\frac{h}{2}(y_2+y_3)+.....+\frac{h}{2}(y_{n-1}+y_n)$$

$$=\frac{h}{2}\left[y_0+y_n+2(y_1+y_2+y_3+...+y_{n-1})\right]$$

The accuracy can always be improved by using more trapezia. Consideration of the curvature in relation to the tops of the trapezia enable us to determine whether our approximation is an over or under estimate.

Mathematical Proof

Disproving a conjecture

To show that a statement is not true it is sufficient to produce just a single COUNTER EXAMPLE.

Ex. Consider the conjecture that $n^2 + n + 1$ is a prime number. Certainly this is true for $n = 1, 2$ and 3

However when $n = 4$ we have $n^2 + n + 1 = 21$ which is not prime. Nothing more needs to be said and the conjecture is false.

Proof by exhaustion

This consists in trying every possible case.

Ex. Prove that if in triangle ABC we have $AB^2 + BC^2 = AC^2$ then ABC is 90^{o}

There are just three possibilities. (i) $ABC < 90^{\mathrm{o}}$, (ii) $ABC = 90^{\mathrm{o}}$, or (iii) $ABC > 90^{\mathrm{o}}$

(iii) If $ABC < 90^{\mathrm{o}}$ then, by the cosine rule in triangle ABC

$AC^2 = AB^2 + BC^2 - 2AB.BC.\cos ABC < AB^2 + BC^2$ since $\cos ABC > 0$

Similarly for case (iii) hence, since one of the three cases must be correct and we have shown that (i) and (iii) are false it follows that (ii) MUST be true.

Typically, this technique consists in exhausting every possibility but one. That remaining one must then be true.

Proof by deduction

In some cases there may be so many possibilities that you cannot try them all. In this case a proof based on a logical sequence of arguments is required. We have already encountered many such proofs earlier in this course. e.g. Proving the formulae for the sums of APs and GPs or proving the Sine and Cosine rules in trigonometry.

Ex. Prove that a triangle with sides $x^2 - 1$, $x^2 + 1$ and $2x$ is right angled.

$(x^2 - 1)^2 + 2x)^2 = x^4 - 2x^2 + 1 + 4x^2 = x^4 + 2x^2 + 1 = (x^2 + 1)^2$

Hence, by the converse of Pythagoras' theorem, the triangle is right angled.

Proof by contradiction

This method depends on the fact that in mathematics it is impossible to start with a true statement and by a logical argument arrive at a false conclusion and involves showing that if a conjecture is not true then an impossible conclusion follows.

Ex. Prove that $\sqrt{2}$ is an irrational number.

Clearly if this statement is not true then $\sqrt{2}$ is a rational number so suppose $\sqrt{2} = \frac{a}{b}$

We can, without loss of generality, assume that $\frac{a}{b}$ is a fraction in its lowest terms and so in particular that a and b have no factor in common.

So we are assuming that $\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \Rightarrow a^2$ is an even number $\Rightarrow a$ is even

So we may write $a = 2k$ for some integer k and so $4k^2 = 2b^2 \Rightarrow b^2 = 2k^2 \Rightarrow b$ is even

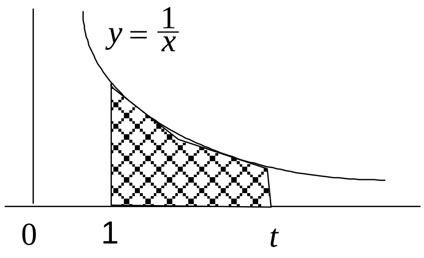
So a and b have a common factor of 2 which contradicts the assumption that $\frac{a}{b}$ was in lowest terms form.

Thus we may conclude that the original assumption was false and hence $\sqrt{2}$ is irrational

Natural Logarithms

Consider the area shaded in the diagram on the right.

Define a function $L(t)$ which measures this area.



Thus $L(t) = \int_1^t \frac{1}{x}\mathrm{d}x$

Consider now $L(ab) = \int_1^{ab} \frac{1}{x}\mathrm{d}x = \int_1^{a} \frac{1}{x}\mathrm{d}x + \int_a^{ab} \frac{1}{x}\mathrm{d}x$

(This should be obvious by considering areas under the graph.)

Now substitute $u = \frac{x}{a}$ so $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{a}$, $x = a \Rightarrow u = 1$ and $x = ab \Rightarrow u = b$ so integral becomes

$\int_1^{b} \frac{1}{au} \times \frac{1}{a}\mathrm{d}u = \int_1^{b} \frac{1}{u}\mathrm{d}u = \int_1^{b} \frac{1}{x}\mathrm{d}x = L(b)$ (Be sre you understand these steps)

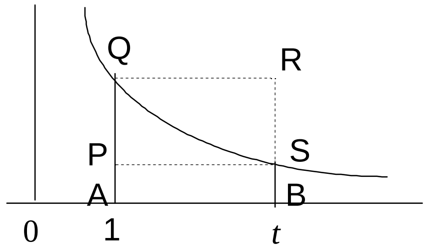
Hence $L(ab) = L(a) + L(b)$ which suggests that L is a logarithmic f unction.

But what is the base of this logarithmic function?

You can find an approximate value for it by carefully drawing the graoh of $y = \frac{1}{x}$ and by counting squares, estimate where t must be for the area to be equal to 1. (since the logarithm of the base is 1)

A more accurate value may be obtained as follows:

Consider the graph of $y = \frac{1}{x}$ on the right.



From the diagram it is clear that

Area ABSP < Area ABSQ < Area ABRQ

i.e. $\frac{1}{t}(t-1) < \int_1^t \frac{1}{x}\mathrm{d}x < (t-1)$

$\Rightarrow \frac{t-1}{t} < L(t) < t-1$, writing $t = 1 + \frac{x}{n}$ this becomes $\frac{x/n}{1+x/n} < L(1+\frac{x}{n}) < \frac{x}{n} \Rightarrow \frac{x}{1+\frac{x}{n}} < L(1+\frac{x}{n})^n < x$

letting $n \to \infty$ we have $\lim_{n\to\infty}\left(\frac{x}{1+\frac{x}{n}}\right) < \lim_{n\to\infty} L(1+\frac{x}{n})^n < x$ and since $\lim_{n\to\infty}\left(\frac{x}{1+\frac{x}{n}}\right) = x$ we have

$\lim_{n\to\infty} L(1+\frac{x}{n})^n = x$

We may assume that $\lim_{n\to\infty} L(1+\frac{x}{n})^n = L\left(\lim_{n\to\infty}(1+\frac{x}{n})^n\right)$ so taking $x = 1$, $L\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right) = 1$

so $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ must be the base of the logarithms. This is usually denoted by e and the function $L(x)$ is more commonly written as $\ln x$ and called the NATURAL (or NAPIERIAN) LOGARITHM.

so $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and the following table shows how the value changes with increasing values of n

n	$\left(1+\frac{1}{n}\right)^n$
2	2.25
10	2.593743
100	2.704814
1,000	2.716924
10,000	2.718204

Further increasing the value of n has no effect on the first 4 significant figures. The accurate value, correct to 8 significant figures is 2.7182818 and despite the apparent recurrence shown here it is an irrational number. 2.718 is sufficiently accurate for most purposes.

Clearly $y = \ln x \Rightarrow x = \mathrm{e}^y$

Functions

You will have heard the word function many times before but most likely without any detailed explanation of what it actually means. Much of the work at A level is directly concerned with the concept of a function and so it is time we defined it more precisely.

We begin with the idea of a BINARY RELATION, or simply a RELATION from a set A to a set B, written $\sigma : A \to B$ which assigns to each ordered pair of elements (a, b) with a in A and b in B exactly one of the statements (i) b is related to a by the relation σ (ii) b is not related to a by the relation σ

For example if σ is the relation 'is greater than' from $A \to B$ where $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$ then 3 is related to 1 by σ, written $3\sigma1$ but 3 is not related to 4 by σ and we write $3\not\sigma4$

A FUNCTION is simply a special kind of relation where there is EXACTLY ONE element of B is related to EACH element of A. Thus, if A and B are non-empty sets, a function from A to B is a rule which associates with EACH member of A, a unique member of B. The set A is the DOMAIN of the function and B is the CO-DOMAIN. Note that A and B can be the same set and note also that it is not necessary to use all the elements of B. For example, if A is the set of real numbers from 0 to 90 inclusive and B is the set of all real numbers, then the trigonometric ration, sine, is a function from A to B that only uses those elements of B between 0 and 1 inclusive.

If we denote the function by f then we write f: $A \to B$ to indicate that f is a function from A to B.. If x is an element of A then we denote by f(x) the unique element of B which f associates with X. f(x) is called the IMAGE of x under f, or the value of f at x. The subset of B consisting of those elements which are images under f of elements of A is called the RANGE of the function.

The most common ways of defining functions are as follows:

(i) A function $\mathrm{f} : ‘ \to ‘$ is defined by $\mathrm{f}(x) = x^2 + 1$ (Note. ‘ is the symbol for the set of real numbers)

(ii) A function f is defined by $\mathrm{f}: x \to \sin x^{\circ}$ for $0 \leqslant x \leqslant 360$

In the second example the co-domain is not stated explicitly but must be a suitable set of real numbers that includes the range of the function.

Functions may be (a) many to one ; this means that two or more elements of the domain may have the same image in the co-domain.

Ex. The second function defined above is an example of this since $\sin 30^{\circ}$ and $\sin 150^{\circ}$ are both equal to 0.5.

(b) one-to-one. This is by far the most important kind of function where each element of the range is the image of exactly one element of the domain. It follows then that exactly one element of the domain is related to each element of the range by some relation or other. And this relation must also be a function. This is called the inverse function and will always exist whenever f is one-to-one. It is denoted by f^{-1}

Thus if f: $‘ \to ‘$ defined by $\mathrm{f}(x) = x + 3$ then $\mathrm{f}^{-1} : ‘ \to ‘$ such that $\mathrm{f}^{-1}(x) = x - 3$.

The easiest way to find an inverse function is to replace x with y say in the original function. Put x equal to this expression and solve for y in terms of x.

Ex.. If $\mathrm{f}: x \to \frac{x}{x-1}$ for $x \in ‘, x \neq 1$ find f^{-1}

The exclusion of – 1 from the domain is necessary because f does not exist for this value of x.

We first require the range of f and since $x - 1$ can take any non-zero value it follows that the range is the complete set ‘ of real numbers.

Next we find the rule for f^{-1}: Put $x = \dfrac{y}{y-1} \Rightarrow xy - x = y \Rightarrow xy - y = x \Rightarrow y(x-1) = x \Rightarrow y = \dfrac{x}{x-1}$

Hence, $\mathrm{f}^{-1}(x) = \frac{x}{x-1}$ for all real x. i.e, this function is SELF-INVERSE.

Ex.. Find $\mathrm{f}^{-1}(x)$ if f:$x \to x^2 + 4x - 5$ for $x \in ‘$

$x^2 + 4x - 5 = (x+2)^2 - 6 \Rightarrow \mathrm{f}(x) \geqslant -6$ for all real x.

$$x = y^2 + 4y - 5 \Rightarrow y^2 + 4y - (x+5) = 0 \Rightarrow y = \frac{-4 \pm \sqrt{16 + 4(x+5)}}{2} = -2 \pm \frac{1}{2}\sqrt{36 + 4x} = -2 \pm \sqrt{9 + x}$$

For this to be a function however we must take only one of the signs in the formula. It is usual to take the positive square root and so $\mathrm{f}^{-1} : x \to -2 + \sqrt{9 + x}$ for $x \geqslant -6$

Clearly, any algebraic function of x can be represented graphically by putting $y = \mathrm{f}(x)$ and plotting corresponding pairs of values of x and y. If (a, b) is a point on this graph then that means that $\mathrm{f}(a) = b$

Hence, by the definition of an inverse function we must also have $\mathrm{f}^{-1}(b) = a$ and so the point (b, a) must lie on the graph of $y = \mathrm{f}^{-1}(x)$ Thus, the graphs of a function and its inverse are symmetrical about the line $y = x$. i.e one is the reflection of the other in the line $y = x$.

The following are important points to note

(a) Each vertical line within the domain must cross the graph at exactly one point.
(b) For an inverse function to exist, each horizontal line within the range must cross the graph exactly once. If necessary, the co-domain should be restricted to satisfy this requirement

Composition of Functions

Let f: $A \to B$ and g: $B \to C$ be functions. With each $a \in A$, the function f associates a unique element $b \in B$ where $b = \mathrm{f}(a)$. Since the domain of g is B, then g associates a unique element $c \in C$ with b such that $c = \mathrm{g}(b)$. Thus $c = \mathrm{g}(b) = \mathrm{g}[\mathrm{f}(a)]$ more usually written simply as $\mathrm{gf}(a)$ This is called the COMPOSITION of f and g , written gf. Note the order of the letters, the right hand one is applied first then the left hand one is applied to that result.

Ex. Let f andg be functions from ‘ → ‘ defined by $\mathrm{f}(x) = x + 1$ and $\mathrm{g}(x) = x^2$
Find gf(0), fg(−1) and find formulae for ff,gg,gf and fg
$\mathrm{gf}(0) = \mathrm{g}(1)$ since $\mathrm{f}(0) = 1$ and $\mathrm{g}(1) = 1$ so $\mathrm{gf}(0) = 1$ $\qquad$ $\mathrm{fg}(-1) = \mathrm{f}(1) = 2$
$\mathrm{ff}(x) = \mathrm{f}(x+1) = x+1+1 = x+2$; $\mathrm{gg}(x) = \mathrm{g}(x^2) = (x^2)^2 = x^4$
$\mathrm{gf}(x) = \mathrm{g}(x+1) = (x+1)^2$; $\mathrm{fg}(x) = \mathrm{f}(x^2) = x^2 + 1$
Note in particular that $\mathrm{fg} \neq \mathrm{gf}$ and this is generally the case.

Some special cases and their graphs.

An EVEN function is one such that $\mathrm{f}(x) = \mathrm{f}(-x)$. The graphs of such functions are necessarily symmetrical about the y axis.

An ODD function is one such that $\mathrm{f}(-x) = -\,\mathrm{f}(x)$. i.e if (a, b) is a point on the graph then so is $(-a, -b)$

The graph thus has rotational symmetry about the origin, with a half turn mapping it back onto itself.

The modulus function. Consider $y = |3x-4|$. $y = 0$ when $x = \frac{4}{3}$ If $x = \frac{4}{3} \pm h$ then $3x - 4 = 4 \pm 3h - 4$ and so $|3x-4| = |\pm 3h| = 3h$ hence, the graph is symmetrical about the line $x = \frac{4}{3}$ This is a common feature of all such modulus graphs. More generally for the graph of $y = |\mathrm{f}(x)|$ we may draw the graph of $y = \mathrm{f}(x)$

And then reflect those parts that lie below the x - axis in the x - axis.

Inequalities involving the modulus function need careful handling.

Ex. $|x-4| \leq 2$ means $x - 4 \leq 2$ or $-(x-4) \leq 2 \Rightarrow x \leq 6$ or $x \geq 2$ so $2 \leq x \leq 6$

Ex. Find the values of x for which $\left|\frac{1}{1+2x}\right| = 1$

$\left|\frac{1}{1+2x}\right| = 1 \Rightarrow \frac{1}{1+2x} = \pm 1 \Rightarrow 1 + 2x = \pm 1 \Rightarrow x = 0$ or -1

Such inequalities can also be solved by squaring both sides first.

$\left|\frac{1}{1+2x}\right| = 1 \Rightarrow \left(\frac{1}{1+2x}\right)^2 = 1 \Rightarrow (1+2x)^2 = 1 \Rightarrow 1 + 2x = \pm 1$ as before.

Note however that if you use this approach you should always check your solutions in case the squaring has introduced spurious ones.

Transformations

You should be familiar with the common transformations from your GCSE work but we consider here the effect that they have on the graph of a function.

(1) The transformation of $y = \text{f}(x)$ represented by $y = a\text{f}(x)$ Clearly for each given value of x, this transformation multiplies the corresponding value of the function by a. Graphically this ‘stretches’ the curve parallel to the y - axis by a factor of ‘a’, points on the x - axis remaining fixed.

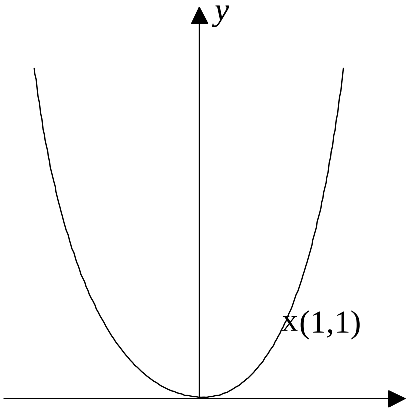
(2) $y = \text{f}(x) + a$. This clearly adds a to each function value and so is a translation of magnitude a parallel to the y - axis.

(3) $y = \text{f}(x - a)$ is a translation of magnitude a parallel to the x - axis

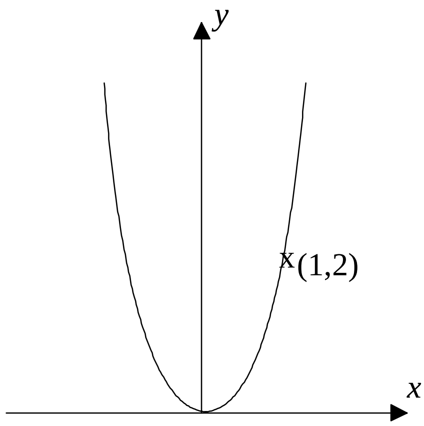
(4) $y = \text{f}(ax)$ is a stretch parallel to the x axis by a factor of $\frac{1}{a}$

Ex. Show the effects on the curve $y = x^2$ of the transformations (a) $y = 2\text{f}(x)$ (b) $y = \text{f}(x) + 2$

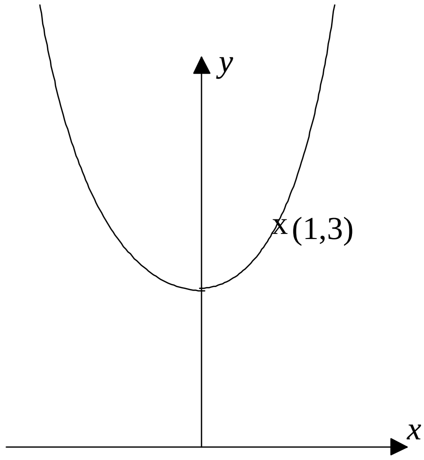
(c) $y = \text{f}(x - 2)$ (d) $y = \text{f}(2x)$ and in each case show the image of the point $P(1, 1)$



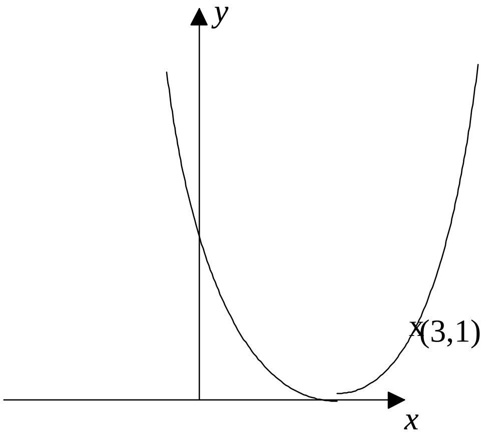
$y = \text{f}(x)$



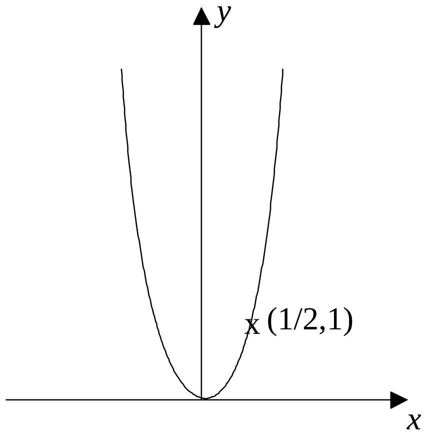
$y = 2\text{f}(x)$



$y = \text{f}(x) + 2$



$y = \text{f}(x - 2)$



$y = \text{f}(2x)$

Further Techniques of Differentiation

So far, if we have had to differentiate $(x+3)^2$, or $\frac{x^2-5}{x}$ or $(x-1)(2x+3)$, we have only been able to if we could multiply or divide out first to reduce it to a polynomial. However this is not always convenient e.g $(3x-2)^{10}$ or $(x-1)^4(2x+3)^5$ or even possible e.g $\frac{x+3}{2x-1}$ Clearly, we need to develop some more rules.

The Chain Rule (function of a function)

Consider $y = (2x^2-3)^5$

If we write $t = 2x^2-3$ then we have $y = t^5$, so y is a function of t which in turn is a function of x. In general, let $y = \mathrm{g}(t)$ where $t = \mathrm{f}(x)$, ie $y = \mathrm{gf}(x)$. If δx is a small increment in the value of x, δt the corresponding change in t and δy the resulting change in y, then we have

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta t} \times \frac{\delta t}{\delta x} \text{ so, letting } \delta x \to 0, \qquad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \mathrm{g}'(t) \times \mathrm{f}'(x)$$

In the above example, $\frac{dy}{dt} = 5t^4$, and $\frac{dt}{dx} = 4x \Rightarrow \frac{dy}{dx} = 5t^4 \times 4x = \underline{20x(2x^2-3)^4}$

Ex. Find $\frac{dy}{dt}$ if $y = \frac{1}{\sqrt{t^2-1}}$

$y = \frac{1}{\sqrt{t^2-1}}$ may be written as $y = (t^2-1)^{-\frac{1}{2}}$ so writing $u = t^2-1$ we have $y = u^{-\frac{1}{2}}$

Hence $\frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$ and $\frac{du}{dt} = 2t$ so $\frac{dy}{dt} = -\frac{t}{2}u^{-\frac{3}{2}} \times 2t = -\frac{t}{u^{\frac{3}{2}}} = \underline{-\frac{t}{\sqrt{(t^2-1)^3}}}$

You should practice until you can write the answers straight down without having to go through all the working.

The Product Rule

Let $y = uv$ where u and v are each functions of x. Then , if δu and δv are the changes in the values of u and v corresponding to a small increment δx in x then we have

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{(u+\delta u)(v+\delta v)-uv}{\delta x}\right) = \lim_{\delta x \to 0}\left(u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u.\frac{\delta v}{\delta x}\right) = u\frac{dv}{dx} + v\frac{du}{dx}$$

The Quotient Rule

Similarly, if $y = \frac{u}{v}$ then $$\frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{\frac{u+\delta u}{v+\delta v} - \frac{u}{v}}{\delta x}\right] = \lim_{\delta x \to 0}\left(\frac{(u+\delta u)v-(v+\delta v)u}{v(v+\delta v)\delta x}\right)$$

$$= \lim_{\delta x \to 0}\left[\frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v(v+\delta v)}\right] = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{v^2}$$

Ex. Differentiate (i) $(x-1)^4(2x+3)^5$ (ii) $\frac{x+3}{2x-1}$

(i) $\frac{d}{dx}\left[(x-1)^4(2x+3)^5\right] = (x-1)^4\frac{d}{dx}(2x+3)^5 + (2x+3)^5\frac{d}{dx}(x-1)^4$

$= (x-1)^4 \times 10(2x+3)^4 + (2x+3)^5 \times 4(x-1)^3$

$= (x-1)^3(2x+3)^4[10(x-1)+4(2x+3)] = \underline{(x-1)^3(2x+3)^4(18x+2)}$

(ii) $\frac{d}{dx}\left(\frac{x+3}{2x-1}\right) = \frac{(2x-1)\frac{d}{dx}(x+3)-(x+3)\frac{d}{dx}(2x-1)}{(2x-1)^2} = \frac{(2x-1)-2(x+3)}{(2x-1)^2} = \underline{\frac{-7}{(2x-1)^2}}$

Inverse Functions

Since $\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$ and $\frac{dx}{dy} = \lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right)$ it should be reasonably obvious that $\frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right)$

This is easily illustrated by considering $y = x^3$ say so that $x = y^{\frac{1}{3}}$

Clearly we have $\frac{dy}{dx} = 3x^2$ and $\frac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}} = \frac{1}{3}(x^3)^{-\frac{2}{3}} = \frac{1}{3x^2}$

Techniques of Integration

1. By Inspection This is certainly the most important and most frequently used method, often involving an intelligent guess followed by a check i.e. differentiating your guess and adjusting if necessary. Certain special patterns should be committed to memory

e.g. Since $\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ whenever we see an integral with $(ax+b)^n$ in the integrand we can guess that the result might be some multiple of $(ax+b)^{n+1}$ More generally, $\frac{d}{dx}[f(x)]^n = [f(x)]^{n-1}f'(x)$

$\Rightarrow \int f'(x)[f(x)]^n dx = A[f(x)]^{n+1} + C$ for some value of A (providing $n \neq -1$)

Ex. Integrate (a) $x(x^2-1)^3$ (b) $\frac{x^2}{\sqrt{1+x^3}}$

(a) Taking $f(x) = x^2 - 1,\ f'(x) = 2x$ As our results can always be adjusted to the extent of a multiplying constant we can 'guess' that the answer is a multiple of $(x^2-1)^4$ and since $\frac{d}{dx}(x^2-1)^4 = 8x(x^2-1)^3$ we see that we must have $\int x(x^2-1)^3 dx = \frac{1}{8}(x^2-1)^4 + C$

(b) $\frac{x^2}{\sqrt{1+x^3}} = x^2(1+x^3)^{-\frac{1}{2}}$ so we guess at $A(1+x^3)^{\frac{1}{2}}$ and since $\frac{d}{dx}(1+x^3)^{\frac{1}{2}} = \frac{3x^2}{2}(1+x^3)^{-\frac{1}{2}}$ we have

$$\int \frac{x^2}{\sqrt{1+x^3}} dx = \frac{2}{3}(1+x^3)^{\frac{1}{2}} + C \text{ or } \frac{2}{3}\sqrt{1+x^3}$$

2. Integration by Substitution

Consider $y = \int f(x)dx \Rightarrow \frac{dy}{dx} = f(x)$ Suppose now we write $x = g(t)$ i.e. put x equal to some function of t.

Then $f(x)$ will become a function of t, $fg(t)$,

Hence $\frac{dy}{dx} = fg(t)$ but $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = fg(t)g'(t) \Rightarrow y = \int fg(t)g'(t)dt$

It is much clearer in practice as can be seen from the following examples

Ex. (ii) By using the substitution $x = 1 + t^2$ find $\int x\sqrt{x-1}\,dx$

$x = 1 + t^2 \Rightarrow \frac{dx}{dt} = 2t$ hence $\int x\sqrt{x-1}\,dx = \int (1+t^2)\sqrt{1+t^2-1}\,2t dt = \int 2t^2(1+t^2)dt$

$$= \int (2t^2 + 2t^4)dt = \frac{2}{3}t^3 + \frac{2}{5}t^5 + C = \frac{2}{3}(x-1)^{\frac{3}{2}} + \frac{2}{5}(x-1)^{\frac{5}{2}} + C$$

A convenient way, in practice, of effecting this change of variable is to note that dx is replaced by $g'(t)dt$and since $\frac{dx}{dt} = g'(t)$ it is just as if $\frac{dx}{dt}$ were a fraction. This technique is particularly useful when the substitution is given in an implicit form. In such a case do not be in too much of a hurry to complete the substitution. Consider what you are putting in place of dx first and you may find some terms will cancel.

Ex. (iii) Use the substitution $1 - 2x^3 = t^2$ to find $\int \frac{x^2}{\sqrt{1-2x^3}} dx$

$1 - 2x^3 = t^2 \Rightarrow -6x^2\frac{dx}{dt} = 2t$ by implicitly differentiating with respect to t so in place of dx we will be putting $-\frac{t}{3x^2}dt$ thus $\int \frac{x^2}{\sqrt{1-2x^3}}dx = \int \frac{x^2}{\sqrt{1-2x^3}}\frac{-t}{3x^2}dt = \int \frac{-t}{3\sqrt{t^2}}dt = \int -\frac{1}{3}dt = -\frac{1}{3}t + C = -\frac{1}{3}\sqrt{1-2x^3} + C$

Differentiation and Integration of Trigonometric Functions

We differentiate $\sin x$ from first principles. Note! x MUST be in radians

An important result that you will learn later on is that $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

$$\frac{d}{dx}(\sin x) = \lim_{\delta x \to 0}\left(\frac{\sin(x+\delta x) - \sin x}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{2\cos\left(x+\frac{1}{2}\delta x\right)\sin\left(\frac{1}{2}\delta x\right)}{\delta x}\right) = \lim_{\delta x \to 0}\left\{\cos\left(x + \frac{1}{2}\delta x\right)\frac{\sin\left(\frac{1}{2}\delta x\right)}{\frac{1}{2}\delta x}\right\} = \cos x$$

This following since $\sin A \approx A$ for small values of A (in radians) so $\frac{\sin\left(\frac{1}{2}\delta x\right)}{\frac{1}{2}\delta x} \to 1$ as $\delta x \to 0$

Similarly we can show that $\frac{d}{dx}(\cos x) = -\sin x$ and by the quotient rule

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Integration

It follows immediately that $\int \cos x\, dx = \sin x + c$, $\int \sin x\, dx = -\cos x + c$ and $\int \frac{1}{\cos^2 x}dx = \tan x + c$

Other trigonometric functions are now easily differentiated by using the above results in conjunction with the product, quotient and function of a function rules.

Ex. $\frac{d}{dx}(\sin 5x) = 5\cos 5x$, $\frac{d}{dx}(\cos^4 x) = -4\cos^3 x \sin x$, $\frac{d}{dx}(\sin 2x \sin x) = \sin 2x \cos x + 2\cos 2x \sin x$
If you learn the patterns of these results you can then integrate other trigonometric functions.

Ex. $\int \cos 3x \, dx = \frac{1}{3}\sin 3x + c$, $\int \cos^2 x \sin x \, dx = -\frac{1}{3}\cos^3 x + c$, $\int \frac{\tan^4 x}{\cos^2 x} dx = \frac{1}{5}\tan^5 x + c$

Differentiation of e^x and $\ln x$

So we now have $\ln x = y \Rightarrow x = e^y$. Natural logarithms and powers of e are of particular importance in calculus. The correct name for the power to which a number is raised is the EXPONENT and variable powers of e, i.e expressions of the form $e^{f(x)}$ are called EXPONENTIAL functions. In particular, e^x is often called THE EXPONENTIAL FUNCTION.

To find the derivative of e^x we note that $\frac{d}{dx}(e^x) = \lim_{h \to 0}\left(\frac{e^{(x+h)} - e^x}{h}\right) = \lim_{h \to 0}\left(\frac{e^x(e^h - 1)}{h}\right)$

If you investigate the behaviour of e^h as $h \to 0$ you should see that it approaches the value of $1 + h$

And so $\lim_{h \to 0}\left(\frac{e^x(e^h - 1)}{h}\right) = \lim_{h \to 0}(e^x) = e^x$ i.e the derivative of e^x is e^x and more generally, $\frac{d(e^{ax})}{dx} = ae^x$

The last result is especially important and must be remembered.

Since $y = e^x \Rightarrow x = \ln y$ then $\frac{dx}{dy} = 1/\frac{dy}{dx} = 1/e^x = 1/y$ i.e. $\frac{d(\ln x)}{dx} = \frac{1}{x}$ This is also a very important result to remember. More generally $\frac{d(\ln ax)}{dx} = \frac{d(\ln a + \ln x)}{dx} = \frac{d(\ln x)}{dx} = \frac{1}{x}$ (since $\ln a$ is constant)

Numerical Solution of Equations

In many real situations we obtain equations which cannot be solved by algebraic or analytical methods. Such equations generally have to be solved by numerical methods which involve obtaining an approximate solution and the refining that solution until we have the required degree of accuracy.
The first step is to obtain an approximate solution. Other than using a graphical calculator to sketch the graph of the function the usual method is to look for a change in sign.
Consider the function $y = \mathrm{f}(x)$. If we draw the graph of this function then to solve the equation $\mathrm{f}(x) = 0$ we require the points at which the graph crosses the x-axis. Clearly as the graph crosses the x-axis the value of the function must change sign, so all we have to do is search for a change in sign, usually by evaluating $\mathrm{f}(x)$ for $x = 0, \pm 1, \pm 2$ etc.
Ex. Locate the roots of $x^5 - 5x + 3 = 0$
Putting $\mathrm{f}(x) = x^5 - 5x + 3$ we have $\mathrm{f}(0) = 3,\ \mathrm{f}(1) = -1,\ \mathrm{f}(2) = 19,\ \mathrm{f}(-1) = 7,\ \mathrm{f}(-2) = -19$
It should be fairly obvious that for $x > 2$ $\mathrm{f}(x)$ will continue to increase and for $x < -2$ $\mathrm{f}(x)$ will continue to decrease so all we can say is that there is at least one root in each of the intervals $[-2, -1]$, $[0, 1]$ and $[1, 2]$.
Note that we can only say 'at least one' but there could possibly be three roots in an interval. Can you explain why there can only be an odd number of roots in any such interval?
Once we have located an interval in which the root lies there are a variety of methods we can use to refine our estimate. These are illustrated with reference to the above equation.

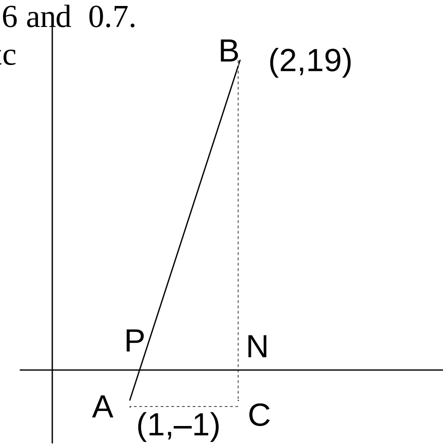
Interval Bisection

Since we know there is a root in the interval . $[-2, -1]$ we try $\mathrm{f}(-1.5)$ which is 2.9 Since this is >0 the root must lie in $[-2, -1.5]$ We next calculate $\mathrm{f}(-1.75) = -4.7$ and so the root is in the interval $[-1.75, -1.5]$ We simply repeat the process as many times as necessary to get the root as accurate as we wish.

Decimal Search

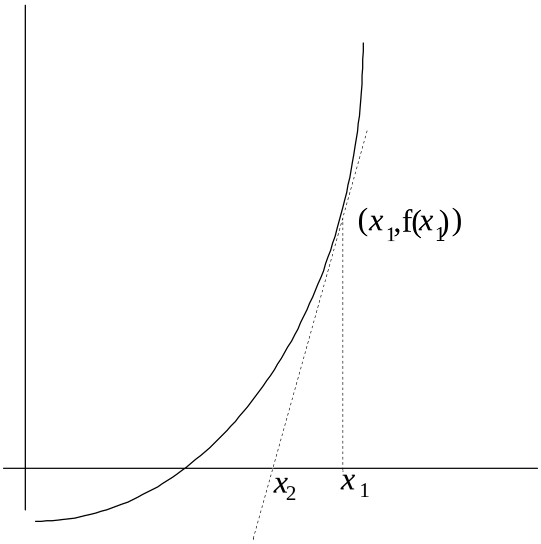
Here we divide our interval into ten parts and repeat the search for a change of sign, so starting with the interval $[0, 1]$ we have $\mathrm{f}(0) = 3,\ \mathrm{f}(0.1) = 2.5,\ \mathrm{f}(0.2) = 2,\ \mathrm{f}(0.3) = 1.5,\ \mathrm{f}(0.4) = 1,\ \mathrm{f}(0.5) = 0.53$
$\mathrm{f}(0.6) = 0.08,\ \mathrm{f}(0.7) = -0.33$ and so the root lies between 0.6 and 0.7.
The process can then be repeated for $x = 0.61,\ 0.62,\ 0.63$, etc

Linear Interpolation



In this method we assume that in the interval we have found the curve can be approximated by a straight line.
Thus, considering the interval $[1, 2]$
We assume a straight line joining the points $(1, -1)$ and $(2, 19)$ as shown and calculate where this line crosses the axis.
By similar triangles, if the x coordinate of P is $1+h$ then $h = \frac{1}{20}$ and so our next estimate for the solution is 1.05
$\mathrm{f}(1.05) = -0.97$ and we could repeat the process for the points $(1.05, -0.97)$ and $(2, 19)$
These methods all come under the heading of 'Change of sign' methods and they do have certain drawbacks.
(1) If the curve touches the x-axis there will be a root (repeated) but no change of sign.
(2) Several roots in the same interval. For example, $x^3 - 1.9x^2 + 1.11x - 0.189 = 0$ has roots at 0.3, 0.7 and 0.9. If you started searching from $x = 0$ you would obtain $x = 0.3$ but might not look any further and so miss the other two roots.
(3) If the curve has a discontinuity you get a change of sign but there is no root.

Fixed Point Estimation



These methods involve working from a single point rather than an interval. There are two main methods.

Iteration

This requires you to rearrange the equation in the form $x = g(x)$ then starting with $x = x_0$ say, we put that into $g(x)$ to give us a new value for x. I.e. We repeatedly apply $x_{n+1} = g(x_n)$ Consider the equation $x^2 - 5 = 0$ This may be rewritten as $2x^2 = x^2 + 5$ and hence $x^2 = \frac{1}{2}(x^2 + 5)$ and finally $x = \frac{1}{2}\left(x + \frac{5}{x}\right)$.

Writing this as $x_{r+1} = \frac{1}{2}\left(x_r + \frac{5}{x_r}\right)$ then by putting $xx_0 = 2$ in the right hand side of this equation we obtain $x_1 = \frac{1}{2} \times 6 = 3$. Repeating the process we have

$x_2 = \frac{1}{2}\left(3 + \frac{5}{3}\right) = 2.333,\ x_3 = \frac{1}{2}\left(2.333 + \frac{5}{2.333}\right) = 2.238,\ x_4 = 2.236,\ x$

and further repetition produces no further change in the value of x_r Hence, $\sqrt{5} = 2.236$

The method can be adopted to calculate any square root by putting that number in place of 5 in the formula the process used here is known as an iterative process each stage being an ITERATION . You will only be required to solve equations by using a given iterative formula.

Ex. Show that $x_{n+1} = \left(x_n + \frac{6}{x_n}\right)^{\frac{1}{2}}$ is an iterative formula for solving the equation $x^3 - x^2 - 6 = 0$. Use this iteration with an initial value of 2 to solve it.

$$x = \left(x + \frac{6}{x}\right)^{\frac{1}{2}} \Rightarrow x^2 = x + \frac{6}{x} \Rightarrow x^3 = x^2 + 6 \Rightarrow x^3 - x^2 - 6 = 0$$

Taking $x_0 = 2$ gives $x_1 = \sqrt{5} = 2.236,\ x_2 = 2.218,\ x_3 = 2.219,\ x_4 = 2.219$

So solution is $\underline{x = 2.219}$ (3 dp)

The Newton-Raphson Method

This involves starting with an estimate x_1 say for the root, you then draw the tangent to the curve at the point with x coordinate x_1 and where this tangent crosses the x-axis gives the next approximation x_2 The process is then repeated as often as necessary.

Theory

The tangent at $(x_1,\ f(x_1))$ has gradient $f'(x_1)$ and so its equation may be written

$$y - f(x_1) = f'(x_1)[x - x_1]$$ and if it crosses the x-axis at $(x_2, 0)$ we must have

$$0 - f(x_1) = f'(x_1)[x_2 - x_1]$$

Rearranging gives $x_2 = x_1 - \dfrac{f(x_1)}{f'x_1)}$ which is really just an iterative formula.

Returning to the example used previously, $x^5 - 5x + 3 = 0$ and taking $x_1 = 0$

We have since, $x_2 = x_1 - \dfrac{x_1^5 - 5x_1 + 3}{5x_1^4 - 5}$ then $x_2 = 0 - \frac{3}{-5} = 0.6$

Repeating $x_3 = 0.6 - \dfrac{0.6^5 - 3 + 3}{5(0.6)^4 - 5} = 0.6 + \dfrac{0.0778}{4.35} = 0.618$

Repeating again $x_4 = 0.618 - \dfrac{0.000145}{-4.27} = 0.61803$

You can see how this method gives a much more rapid rate of convergence than any of the methods discussed previously and though it can fail, for example if the function is discontinuous or if the root of $f(x) = 0$ is also an approximate root of $f'(x) = 0$, it is usually the best method to use.

General Binomial Expansion

You have previously met the binomial expansion

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{r}x^r + \ldots + x^n$$

Which holds for any positive integer n.

Let us now assume that $(1+x)^n$ can be written in the form of a series in ascending powers of x for other values of n.

So assume that $(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r +$

Writing $f(x) = (1+x)^n$ we have $f(0) = 1 \Rightarrow a_0 = 1$

$f'(x) = n(1+x)^{n-1} = a_1 + 2a_2x + 3a_3x^2 + \ldots + ra_rx^{r-1} + \ldots$ and $f'(0) = n$ so $a_1 = n$

$f''(x) = n(n-1)(1+x)^{n-2} = 2a_2 + 2.3a_3x + 3.4a_4x^2 + \ldots. + r(r-1)a_rx^{r-2} + \ldots$ so $a_2 = \frac{n(n-1)}{2}$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} = 2.3a_3 + 2.3.4a_4x + 3.4.5a_5x^2 + \ldots$ so $a_3 = \frac{n(n-1)(n-2)}{2.3}$

If you study the pattern of what is happening as we continue to differentiate you should be able to see that in general, $a_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = \binom{n}{r}$ and we see that the series is exactly the same as it was for positive integral values of n i.e.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

Note now that since n is not a positive integer, the coefficients will never be zero and so the series will be infinite. It is therefore necessary that it should be convergent. The study of convergence is beyond the scope of the course but it can be shown that the series converges providing $|x| < 1$, i.e. $-1 < x < 1$.

Ex. Find the first four terms in the expansions of (a) $(1+x)^{-1}$ (b) $(1+x)^{-2}$ (c) $(1-x)^{-1}$ (d) $(4-3x)^{1/2}$

(a) $(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \ldots = 1 - x + x^2 - x^3$ This is worth trying to learn as a special case.

(b) $(1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \ldots = 1 - 2x + 3x^2 - 4x^3 + \ldots$

(c) $(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \ldots = 1 + x + x^2 + x^3 + ..$ This is also worth committing to memory if you can.

(d) $(4-3x)^{1/2}$ the problem here is that the general series obtained above only applies for a power of"one plus an x term" so we must put it in this form first, thus:

$$(4-3x)^{1/2} = 4^{1/2}(1 - \frac{3x}{4})^{1/2} = 2\left(1 + \frac{1}{2}\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(-\frac{3x}{4}\right)^3 + \ldots\right)$$

$$= 2\left(1 - \frac{3x}{8} - \frac{9x^2}{128} - \frac{27x^3}{1024} + \ldots\right) \text{ for } \left|\frac{3x}{4}\right| < 1 \Rightarrow |x| < \frac{4}{3}$$

Note that we may now write an even more general form of the binomial expansion;

$$(a+b)^n = a^n\left(1 + \frac{b}{a}\right)^n = a^n\left(1 + n\frac{b}{a} + \frac{n(n-1)}{2}\left(\frac{b}{a}\right)^2 + \ldots\right) = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots$$

and since we require the x term to be numerically less than one, the condition for convergence now becomes $\left|\frac{b}{a}\right| < 1$

Ex. Estimate $\sqrt[3]{997}$ correct to 6 d.p.

<u>Solution</u>. $997 = 1000 - 3$ and 1000 is a perfect cube so $(997)^{1/3} = (1000-3)^{1/3} = 10(1-0.003)^{1/3}$

Hence, $\sqrt[3]{997} = 10\left(1 + \frac{1}{3}(-0.003) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}(0.003)^2 + \ldots\right)$

$= 10(1 - 0.001 - 0.000001\ldots)$ it is clear that subsequent terms will not affect the 6th dp.

So $\sqrt[3]{997} = 10 \times 0.9989990 = 9.989990$

Rational Functions

i.e. Functions of the form $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials in x.

Simplifying

We simplify algebraic fractions in the same way as arithmetic fractions, i.e. By dividing numerator and denominator by the same factor.

Ex. $\frac{x^3+2x^2-3x}{2x^3+10x^2+6x}$ we first factorise numerator and denominator completely.

i.e. $\frac{x(x^2+2x-3)}{2x(x^2+5x+6)} = \frac{x(x-1)(x+3)}{2x(x+2)(x+3)}$ and then divide numerator and denominator by the common factors x and $x+3$ to finish up with $\frac{x-1}{x+2}$

This is how it would look in practice $\frac{x^3+2x^2-3x}{2x^3+10x^2+6x} = \frac{x(x^2+2x-3)}{2x(x^2+5x+6)} = \frac{x(x-1)(x+3)}{2x(x+2)(x+3)} = \frac{x-1}{x+2}$

Addition/Subtraction

We add or subtract two such functions by expresssing them with a common denominator exactly as in adding/subtracting arithmetic fractions.

Ex. . $\frac{3}{x(x+1)} + \frac{2x}{(x+1)^2(x-1)}$ we have a common denominator of $x(x+1)^2(x-1)$ so expressing each fraction with this denominator we have $\frac{3(x-1)(x+1)}{x(x+1)^2(x-1)} + \frac{2x^2}{x(x+1)^2(x-1)} = \frac{2x^2+3x^2-3}{x(x+1)^2(x-1)} = \frac{5x^2-3}{x(x+1)^2(x-1)}$

Multiplication/Division

Again the procedure is exactly as in ordinary arithmetic, but always be on the lookout for the possibility of cancelling common factors in numerators and denominators.

Ex. $\frac{2x+4}{x^2-1} \times \frac{3x-3}{x^2-4} = \frac{2(x+2)}{(x+1)(x-1)} \times \frac{3(x-1)}{(x+2)(x-2)} = \frac{2}{x+1} \times \frac{3}{x-2} = \frac{6}{(x+1)(x-2)}$

Ex. $\frac{x^2-9}{x^2+1} \div \frac{x^2-x-6}{x^3+x} = \frac{(x+3)(x-3)}{x^2+1} \times \frac{x(x^2+1)}{(x-3)(x+2)} = \frac{x+3}{1} \times \frac{x}{x+2} = \frac{x(x+3)}{x+2}$

Partial Fractions

From elementary algebra we know that $\frac{2}{x-3} - \frac{3}{x-2} = \frac{2(x-2)-3(x-3)}{(x-2)(x-3)} = \frac{5-x}{(x-2)(x-3)}$

We are now going to consider the reverse of this process, i.e splitting a rational function into the sum or difference of two or more simpler fractions or PARTIAL FRACTIONS. Note the following:-

(a) the denominator of the single fraction is the L.C.M. of the denominators of the partial fractions.

(b) if the numerator of each partial fraction is of lower degree than its denominator, then this must also apply to the combined fraction and conversely.

We may therefore deduce that there will be a partial fraction corresponding to each distinct factor of the denominator of a rational function and, providing the numerator of this function is of lower degree than its denominator , then this will also apply to each partial fraction. In particular to each linear factor $ax+b$ in the denominator there will correspond a partial fraction of the form $\frac{A}{ax+b}$ with A constant.

Ex. Express $\frac{x+18}{(x-3)(2x+1)}$ in partial fractions.

Let $\frac{x+18}{(x-3)(2x+1)} = \frac{A}{x-3} + \frac{B}{2x+1} = \frac{A(2x+1)+B(x-3)}{(x-3)(2x+1)}$ so we require $A(2x+1)+B(x-3) \equiv x+18$

In particular when $x=3$ we have $7A = 21 \Rightarrow A = 3$ and when $x = -\frac{1}{2}$ then $-\frac{7}{2}B = 17\frac{1}{2} = \frac{35}{2} \Rightarrow B = -5$

Hence, $\frac{x+18}{(x-3)(2x+1)} = \frac{3}{x-3} - \frac{5}{2x+1}$

This method can in fact be shortened by noting that from $A(2x+1)+B(x-3) \equiv x+18$ by letting $x=3$ we have $A = \frac{x+18}{2x+1}$ evaluated at $x=3$ whilst putting $x=-\frac{1}{2}$ we have $B = \frac{x+18}{x-3}$ evaluated at $x = -\frac{1}{2}$

This gives us the COVER-UP RULE. "To find the numerator for a factor $x-k$, we 'cover-up' $x-k$ in the original function and put $x=k$ in the rest of this function."

Ex. $\frac{4x^2-19x+7}{(x-1)(x-2)(x+3)} = \frac{1}{x-1} \times \frac{4-19+7}{(-1)\times 4} + \frac{1}{x-2} \times \frac{16-38+7}{1\times 5} + \frac{1}{x+3} \times \frac{36+57+7}{(-4)\times(-5)} = \frac{2}{x-1} - \frac{3}{x-2} + \frac{5}{x+3}$

Ex. Express $\frac{x^2+x}{x^2-4}$ in partial fractions

There is a problem here in that the numerator is not of lower degree than the denominator so we must do one stage of division first to give $\frac{x^3+x}{x^2-4} = \frac{x(x^2-4)+5x}{x^2-4} = x + \frac{5x}{x^2-4}$

We can now complete the solution using the cover-up rule

$$\frac{x^3+x}{x^2-4} = x + \frac{5x}{(x-2)(x+2)} = x + \frac{5}{2(x-2)} + \frac{5}{2(x+2)}$$

Sometimes we might have a repeated factor in the denominator, e.g $\frac{2x^2-3}{(x-1)^3(x+1)}$

Thinking again of the process of combining fractions, it should be clear that we MUST have a partial fraction with a denominator of $(x-1)^3$ as otherwise it would not appear in the final denominator. However we MIGHT also have partial fractions with denominators $(x-1)$ and $(x-1)^2$. By the same arguments as before, all of these fractions would have constant numerators.

Thus, $\frac{2x^2-3}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}$

$\Rightarrow 2x^2 - 3 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3$

Putting $x = 1$ gives $-1 = 2C$ so $C = -\frac{1}{2}$

Putting $x = -1$ gives $-1 = -8D$ so $D = \frac{1}{8}$

Comparing coefficients of x^3 we have $0 = A + D \Rightarrow A = -\frac{1}{8}$

Comparing constant terms we have $-3 = A - B + C - D$ so $B = 3 + A + C - D = \frac{9}{4}$

Hence, $\frac{2x^2-3}{(x-1)^3(x+1)} = \frac{1}{8(x+1)} - \frac{1}{8(x-1)} + \frac{9}{4(x-1)^2} - \frac{1}{8(x-1)^3}$

Note! Careful choice of method will minimise the amount of work you have to do.

The final problem is that we might have an irreducible quadratic factor in the denominator. From the foregoing theory it should be reasonably obvious that we can no longer assume a constant numerator, only that the numerator is of lower degree than the denominator. We must therefore allow for a linear numerator in such cases.

Ex. $\frac{x^3-2}{x^4-1} = \frac{x^3-2}{(x^2+1)(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$

So we require $x^3 - 2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)$

Putting $x = 1$ gives $-1 = 4B$ so $B = -\frac{1}{4}$, putting $x = -1$ gives $-3 = -4A$ so $A = \frac{3}{4}$

Comparing coefficients of x^3 gives $1 = A + B + C \Rightarrow C = 1 - A - B = \frac{1}{2}$

Whilst comparing constant terms gives $-2 = -A + B - D \Rightarrow D = 2 - A + B = 1$

Hence, $\frac{x^3-2}{x^4-1} = \frac{3}{4(x+1)} - \frac{1}{4(x-1)} + \frac{\frac{1}{2}x+1}{x^2+1} = \frac{3}{4(x+1)} - \frac{1}{4(x-1)} + \frac{x+2}{2(x^2+1)}$

Note that comparing constant terms is equivalent to putting $x = 0$

One important use of partial fractions is to obtain an expansion for a complicated fraction.

Ex. Expand $\frac{x^3-2}{x^4-1}$ as a series of ascending powers of x. as far as the term in x^3

Solution. From the previous example we have

$$\begin{aligned}\frac{x^3-2}{x^4-1} &= \frac{3}{4(x+1)} - \frac{1}{4(x-1)} + \frac{x+2}{2(x^2+1)} = \frac{3}{4}(1+x)^{-1} + \frac{1}{4}(1-x)^{-1} + \frac{1}{2}(x+2)(1+x^2)^{-1} \\ &= \frac{3}{4}(1 - x + x^2 - x^3 \ldots) + \frac{1}{4}(1 + x + x^2 + x^3 \ldots) + \frac{1}{2}(x+2)(1 - x^2 \ldots) \\ &= 1 - \frac{1}{2}x + x^2 - \frac{1}{2}x^3 \ldots + 1 + \frac{1}{2}x - 2x^2 - \frac{1}{2}x^3 = 2 - x^2 - x^3\end{aligned}$$

More Trigonometry

Reciprocal functions

As well as the main trigonometric functions $\sin x,\ \cos x$ and $\tan x$ we often have a need for their reciprocals, which are defined by $\sec x = \frac{1}{\cos x}$ (the secant of x), $\operatorname{cosec} x = \frac{1}{\sin x}$ (the cosecant of x) and $\cot x = \frac{1}{\tan x}$ (the cotangent of x)

Note that each of these is undefined for certain values of x, e.g. $\sec x$ is undefined for $x = n\pi$, since $\cos n\pi = 0$, $\operatorname{cosec} x$ is undefined for $x = (2n-1)\frac{\pi}{2}$ and $\cot x$ is undefined for $x = n\pi$ since $\sin x$ and $\tan x$ are zero at these values.

Addition Theorems

Consider a positive rotation through an angle θ about the origin. From our pre A-level work we know that this can be represented by the matrix $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ similarly, a rotation through angle ψ may be represented by the matrix $\mathbf{B} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$

The matrix product $\mathbf{AB}$ will thus represent a rotation through angle $\theta+\psi$. But this rotation is also represented by the matrix $\begin{pmatrix} \cos(\theta+\psi) & -\sin(\theta+\psi) \\ \sin(\theta+\psi) & \cos(\theta+\psi) \end{pmatrix}$ so equating these two representations we have

$$\begin{pmatrix} \cos(\theta+\psi) & -\sin(\theta+\psi) \\ \sin(\theta+\psi) & \cos(\theta+\psi) \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\psi - \sin\theta\sin\psi & -\cos\theta\sin\psi - \sin\theta\cos\psi \\ \sin\theta\cos\psi + \cos\theta\sin\psi & -\sin\theta\sin\psi + \cos\theta\cos\psi \end{pmatrix}$$

And comparing corresponding elements we have $\sin(\theta+\psi) = \sin\theta\cos\psi + \cos\theta\sin\psi$

and $\cos(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$

Similarly by considering a rotation of θ followed by one of $-\psi$, or by replacing ψ with $-\psi$ in the preceding results we have also $\sin(\theta-\psi) = \sin\theta\cos\psi - \cos\theta\sin\psi$

and $\cos(\theta-\psi) = \cos\theta\cos\psi + \sin\theta\sin\psi$

To find corresponding results for tangents we proceed as follows:

$\tan(\theta+\psi) = \frac{\sin(\theta+\psi)}{\cos(\theta+\psi)} = \frac{\sin\theta\cos\psi + \cos\theta\sin\psi}{\cos\theta\cos\psi - \sin\theta\sin\psi} = \frac{\tan\theta + \tan\psi}{1-\tan\theta\tan\psi}$ the last form being obtained by dividing numerator and denominator by $\cos\theta\cos\psi$

Similarly $\tan(\theta-\psi) = \frac{\sin(\theta-\psi)}{\cos(\theta-\psi)} = \frac{\sin\theta\cos\psi - \cos\theta\sin\psi}{\cos\theta\cos\psi + \sin\theta\sin\psi} = \frac{\tan\theta - \tan\psi}{1+\tan\theta\tan\psi}$

These six identities are known collectively as the addition theorems and should be committed to memory.

Ex. Express $\sin 15^{\circ}$ in surd form

$$\sin 15^{\circ} = \sin(45^{\circ} - \sin 30^{\circ}) = \sin 45^{\circ}\cos 30^{\circ} - \cos 45^{\circ}\sin 30^{\circ} = \frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}.\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Ex. If $\sin x = \frac{5}{13}$ and $\sin y = \frac{4}{5}$ and x, y are both obtuse angles, find $\cos(x+y)$

$\cos x = -\sqrt{(1-\sin^2 x)} = -\sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = -\frac{12}{13}$, similarly $\cos y = -\frac{3}{5}$

Hence, $\cos(x+y) = \cos x\cos y - \sin x\sin y = \frac{36}{65} - \frac{20}{65} = \frac{16}{65}$

Ex. Expand $\sin(x+y+z)$

$$\begin{aligned}\sin(x+y+z) = \sin(x+\{y+z\}) &= \sin x\cos(y+z) + \cos x\sin(y+z) \\ &= \sin x(\cos y\cos z - \sin y\sin z) + \cos x(\sin y\cos z + \cos y\sin z) \\ &= \sin x\cos y\cos z + \cos x\sin y\cos z + \cos x\cos y\sin z - \sin x\sin y\sin z\end{aligned}$$

Ex Prove that $\tan x + \tan y \equiv \sin(x+y)\sec x\sec y$

$$\tan x + \tan y = \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y} = \frac{\sin(x+y)}{\cos x\cos y} = \sin(x+y)\sec x\sec y$$

Ex Find the acute angle between the straight lines $y = 2x+1$ and $y = 3x-2$

Let lines make angles α and β respectively with $x-$ axis then the angle between the lines is $\theta = \beta - \alpha$ (verify this for yourself by sketching a diagram) Now $\tan\alpha = 2$ and $\tan\beta = 3$ and so

$$\tan\theta = \tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} = \frac{3-2}{1+6} = \frac{1}{7} \Rightarrow \theta = 8.1^{\circ}$$

Double/Half angle identities

Putting $\psi = \theta$ in the addition theorems immediately gives us
$\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
The identity for $\cos 2\theta$ has two important alternative forms obtained by making use of the identity
$\sin^2\theta + \cos^2\theta = 1$, thus we have $\cos 2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$
and $\cos 2\theta = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$
These results provide the only means of integrating $\sin^2\theta$ and $\cos^2\theta$ by rearranging them in the form $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and so are extremely important.

Ex. Express $\sin 3x$ in terms of $\sin x$
$\sin 3x = \sin(x+2x) = \sin x\cos 2x + \cos x\sin 2x = \sin x(1-2\sin^2 x) + \cos x(2\sin x\cos x)$
$= \sin x - 2\sin^3 x + 2\sin x\cos^2 x = \sin x - 2\sin^3 x + 2\sin x(1-\sin^2 x) = 3\sin x - 4\sin^3 x$

Ex. Solve the equation $\sin 2\theta = \cos\theta$ giving all solutions between 0^o and 360^o inclusive.
$\sin 2\theta = \cos\theta \Rightarrow 2\sin\theta\cos\theta = \cos\theta \Rightarrow \cos\theta = 0$ or $\sin\theta = \frac{1}{2}$
$\cos\theta = 0 \Rightarrow \theta = 90^o$ or 270^o and $\sin\theta = 0.5 \Rightarrow \theta = 30^o$ or 150^o so $\theta = 30^o, 90^o, 150^o$ or 270^o
NOTE! Beware of falling into the trap of dividing through by $\cos\theta$ and so losing two solutions

Ex. Prove that $\frac{\cos 2x}{1-\sin 2x} \equiv \frac{1+\tan x}{1-\tan x}$
$\frac{\cos 2x}{1-\sin 2x} = \frac{\cos^2 x - \sin^2 x}{1 - 2\sin x\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x - 2\sin x\cos x} = \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\sin x - \cos x)^2} = \frac{\cos x + \sin x}{\sin x - \cos x}$
$= \frac{1+\tan x}{1-\tan x}$ by dividing top and bottom by $\cos x$

Factor Theorems.

From $\sin(x+y) = \sin x\cos y + \cos x\sin y$ and $\sin(x-y) = \sin x\cos y - \cos x\sin y$ we have, by adding $\sin(x+y) + \sin(x-y) = 2\sin x\cos y$ and writing $x+y = A$ and $x-y = B$ this becomes
$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$ This is the first of the factor theorems, so called because it enables us to express the sum of two sines as a product of two factors. It is best remembered in words. “The sum of two sines is twice the product of the sine of half the sum and the cosine of half the difference”.
Similarly $\sin A + \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$, $\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A\_B)$ and,
$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$. Notice the negative sign in the last one.

Ex. Factorise $\sin 3\theta + \sin\theta$
$\sin 3\theta + \sin\theta = 2\sin\frac{1}{2}(3\theta+\theta)\cos\frac{1}{2}(3\theta-\theta) = 2\sin 2\theta\cos\theta$

Ex. Evaluate $\sin 75^o - \sin 15^o$ without using tables or a calculator
By factor theorem $\sin 75^o - \sin 15^o = 2\cos 45^o\sin 30^o = 2\times\frac{\sqrt{2}}{2}\times\frac{1}{2} = \frac{\sqrt{2}}{2}$

Ex. Prove that $\frac{\sin 5x - \sin x}{\cos 4x + \cos 2x} \equiv 2\sin x$
$\frac{\sin 5x - \sin x}{\cos 4x + \cos 2x} = \frac{2\cos 3x\sin 2x}{2\cos 3x\cos x} = \frac{\sin 2x}{\cos x} = \frac{2\sin x\cos x}{\cos x} = 2\sin x$

Ex Solve $\sin 5x + \sin x = \sin 3x$ for $0^o \le x \le 180^o$
$\sin 5x + \sin x = \sin 3x \Rightarrow 2\sin 3x\cos 2x = \sin 3x \Rightarrow \sin 3x = 0$ or $\cos 2x = 0.5$
$\sin 3x = 0 \Rightarrow 3x = 0^o, 180^o, 360^o$ or $540^o \Rightarrow x = 0^o, 60^o, 120^o$ or 180^o
and $\cos 2x = 0.5 \Rightarrow 2x = 60^o$ or $300^o \Rightarrow x = 30^o$ or 150^o so $x = 0^o, 30^o, 60^o, 120^o, 150^o$ or 180^o

$a\cos x + b\sin x$

We frequently encounter expressions of the above form. If $a = b$ we can use the factor theorems as follows
Ex. $2\cos x + 2\sin x = 2\sin(90^o - x) + 2\sin x = 2\sin 45^o\cos(45^o - x) = \sqrt{2}\cos(45^o - x)$
More often though we have $a \neq b$ so the factor theorems cannot be used. We then proceed as follows:
Ex. $3\sin x + 4\cos x$ may be written as $R\sin(x+\theta) = R\sin x\cos\theta + R\cos x\sin\theta$ if we have
$R\cos\theta = 3$ and $R\sin\theta = 4 \Rightarrow R^2 = 3^2 + 4^2 = 25$ ie $R = 5$ and $\tan\theta = \frac{4}{3} \Rightarrow \theta = 53.1^o$
So $3\sin x + 4\cos x = 5\sin(x + 53.1^o)$
More generally, $a\sin x + b\cos x = R\sin(x+\alpha)$ with $R = \sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$

It is conventional to take $R > 0$ and by carefully choosing the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$ one can ensure that α is acute. The general patterns are as follows

$$a\sin x + b\cos x = R\sin(x+\alpha) \qquad a\cos x + b\sin x = R\cos(x-\alpha)$$
$$a\sin x - b\cos x = R\sin(x-\alpha) \qquad a\cos x - b\sin x = R\cos(x+\alpha)$$

where in each case $R = \sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$. Be sure you understand the above patterns.

An important application is to obtain the maximum and minimum values of expressions such as $a\cos x + b\sin x$. Since $\sin(x \pm \alpha)$ and $\cos(x \pm \alpha)$ take values between 1 and –1 inclusive it follows that the maximum and minimum values of such expressions are R.and–R respectively .

The other important application is in solving equations

Ex. Solve $5\cos\theta - 6\sin\theta = 4$ for $0^{\circ} \le \theta \le 360^{\circ}$

We have $\sqrt{5^2+6^2}\cos(\theta+\alpha) = 4 \Rightarrow \cos(\theta+\alpha) = \frac{4}{\sqrt{61}} = 0.5121$ with $\alpha = \tan^{-1}\left(\frac{6}{5}\right) = 50.2^{\circ}$

$0^{\circ} \le \theta \le 360^{\circ} \Rightarrow 50.20^{\circ} \le \theta + 50.20^{\circ} \le 410.2$ and $\cos^{-1}0.5121 = 59.2^{\circ}$

So $\theta + 50.2^{\circ} = 59.2^{\circ}$ or $300.8^{\circ} \Rightarrow \theta = 9^{\circ}$ or 250.6°

Parametric Co-ordinates

With some curves it is often convenient to consider x and y as functions of a third variable, known as a PARAMETER.

Ex 1. The equation $y^2 = 4x$ is satisfied by $x = t^2,\ y = 2t$ for all values of t. and so $x = t^2,\ y = 2t$ may be used to describe the curve instead of the cartesian equation $y^2 = 4x$ they are called PARAMETRIC EQUATIONS of the curve or PARAMETRIC CO-ORDINATES of a point on the curve.

Ex. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ has parametric equations $x = a\cos^3\theta,\ y = a\sin^3\theta$

Elimination of the parameter from a pair of parametric equations will produce the corresponding cartesian equation.

Note particularly that the parametric equations of a circle of radiue a are $x = a\cos\theta,\ y = a\sin\theta$

Ex. If $x = 3t - 1$ and $y = 2t^2 + 1$ then eliminating t we have $y = 2\left(\frac{x+1}{3}\right)^2 + 1$

$$\Rightarrow 9y = 2(x+1)^2 + 9 \text{ or } 9y = 2x^2 + 4x + 11$$

We may often require to find the gradient of a curve defined by parametric equations and even if it is possible to obtain the cartesian form it may be extremely difficult to differentiate so we need to be able to find $\frac{dy}{dx}$ directly from the parametric equations.

We first of all note that, from the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Ex. Find $\frac{dy}{dx}$ if $x = 3t^2 - t$ and $y = t^3$.

Solution We have $\frac{dx}{dt} = 6t - 1$ and $\frac{dy}{dt} = 3t^2$ hence $\frac{dy}{dx} = \frac{3t^2}{6t-1}$

Ex. Find the equation of the tangent to the curve defined by $x = 2t^2,\ y = 4t$ at the point where $t = 2$

Solution When $t = 2,$ we have $x = 8,\ y = 8.$ $\frac{dy}{dt} = 4$ and $\frac{dx}{dt} = 4t$ so $\frac{dy}{dx} = \frac{4}{4t} = \frac{1}{t} = \frac{1}{2}$ when $t = 2$

Equation of tangent is thus $y - 8 = \frac{1}{2}(x - 8)$ or $2y - 16 = x - 8 \Rightarrow x - 2y + 8 = 0$

Further Calculus

Volume of Revolution

If the area between a curve and an axis is rotated through 360^{o}
about that axis then it generates a solid of revolution. The volume of such a solid may be found as follows:

Volume generated by a typical element (shown shaded) $= \pi y^2 \delta x$

hence, total volume is $\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x = \int_a^b \pi y^2 \mathrm{d}x$

Ex Find the volume generated when the area between the curve $y = x^2$, the $x-$ axis and the ordinates $x = 1,\ x = 2$ is rotated through 4 right angles about the x -axis

Before we can integrate we must express y^2 as a function of x.

we then have volume $= \int_1^2 \pi x^4 \mathrm{d}x = \pi\left[\frac{1}{5}x^5\right]_1^2 = \pi\left(\frac{32}{5} - \frac{1}{5}\right) = \frac{31\pi}{5}$

Ex. Find the volume generated by the rotation of the area between the curve $y = x^2$, the $y-$ axis and the lines $y = 1,\ y = 4$.

Interchanging the roles of x and y we have volume $= \int_1^4 \pi x^2 \mathrm{d}y = \pi \int_1^4 y \mathrm{d}y = \pi\left[\frac{1}{2}y^2\right]_1^4$

$$= \pi\left(8 - \frac{1}{2}\right) = \frac{15\pi}{2}$$

Note that it is normal practice to take the π outside the integral since it is a constant.

Use of Partial Fractions in Integration

Rational functions can often be integrated by first expressing them in partial fractions.

Ex. Integrate $\frac{4x^2-19x+7}{(x-1)(x-2)(x+3)}$

so $\int \frac{4x^2-19x+7}{(x-1)(x-2)(x+3)}\mathrm{d}x = \int\left(\frac{2}{x-1} - \frac{3}{x-2} + \frac{5}{x+3}\right)\mathrm{d}x = 2\ln(x-1) - 3\ln(x-2) + 3\ln(x+3) + C$

It is often better in a case such as this to write the constant of integration as $\ln A$ rather than C

giving the result $\ln(x-1)^2 - \ln(x-2)^3 + \ln(x+3)^3 + \ln A = \ln \frac{A(x-1)^2(x+3)^3}{(x-2)^3}$

Ex Evaluate $\int_2^3 \frac{2x^2-3}{(x-1)^3(x+1)}\mathrm{d}x$

$\frac{2x^2-3}{(x-1)^3(x+1)} = \frac{1}{8(x+1)} - \frac{1}{8(x-1)} + \frac{9}{4(x-1)^2} - \frac{1}{8(x-1)^3}$ (By standard techniques)

so $\int_2^3 \frac{2x^2-3}{(x-1)^3(x+1)}\mathrm{d}x = \int_2^3\left(\frac{1}{8(x+1)} - \frac{1}{8(x-1)} + \frac{9}{4(x-1)^2} - \frac{1}{8(x-1)^3}\right)\mathrm{d}x$

$= \left[\frac{1}{8}\ln(x+1) - \frac{1}{8}\ln(x-1) - \frac{9}{4(x-1)} - \frac{1}{16(x-1)^2}\right]_2^3 = \left(\frac{1}{8}\ln 4 - \frac{1}{8}\ln 2 - \frac{9}{8} - \frac{1}{64}\right) - \left(\frac{1}{8}\ln 3 - \frac{9}{4} - \frac{1}{16}\right)$

$= \frac{1}{8}\ln 2 - \frac{1}{8}\ln 3 - \frac{72+1-144-4}{64} = \frac{1}{8}\ln\frac{2}{3} + \frac{75}{64}$

Differential Equations

These are equations relating two variables and one or more derivatives of one variable with respect to the other. They occur in the mathematical description of a wide variety of physical situations.

Ex. Newton's Law of Cooling $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_0)$

Simple Harmonic Motion $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x$

An Electrical Circuit $L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{k} = E$

Differential equations are classified by order and degree. The ORDER is that of the highest order derivative in the equation. The DEGREE is the power of the highest order derivative. So the first example above is first order first degree whilst the other two are second order first degree.

Differential equations may also be used in geometry to describe families of curves. They are formed by the elimination of arbitrary constants by repeated differentiation. In general, n differentiations are required to eliminate n arbitrary constants and produce a differential equation of order n.

Ex. Form differential equations to describe the following families of curves.

(a) Circles with centre at the origin, (b) parabolas with vertical axis

(c) $y = \mathrm{e}^{-t}(A\sin 2t + B\cos 2t)$

(a) We have $x^2 + y^2 = r^2$ (r constant) so differentiating with respect to x

$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ or $y\frac{\mathrm{d}y}{\mathrm{d}x} + x = 0$ which is the required differential equation. It tells us that the tangent is perpendicular to the radius, which is a defining property of a circle.

(b) The general equation is $y = ax^2 + bx + c$ so differentiating repeatedly we have

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b$, (ii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2a$ and (iii) $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0$ which is the required differential equation

(c) $y = \mathrm{e}^{-t}(A\sin 2t + B\cos 2t) \Rightarrow \mathrm{e}^{-t}y = (A\sin 2t + B\cos 2t)$ so differentiating

$$\mathrm{e}^{-t}\frac{\mathrm{d}y}{\mathrm{d}t} - \mathrm{e}^{-t}y = 2A\cos 2t - 2B\sin 2t \text{ and differentiating again}$$

$$\mathrm{e}^{-t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\mathrm{e}^{-t}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{-t}y = -4A\sin 2t - 4B\cos 2t = -4\mathrm{e}^{-t}y$$

Dividing by e^{-t} which we may do since it cannot be zero we have $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = 0$

As has already been stated, a great variety of physical situations are modeled by differential equations. If we wish to find a direct relationship between the two variables then we must 'solve' the differential equation. To do this we must eliminate the derivatives by some form of integration. Each integration introduces an arbitrary constant and the 'general solution' or 'complete primitive' of a differential equation of order n must contain n arbitrary constants. If sufficient initial (or other) conditions are given then it is possible to evaluate these constants and obtain a particular solution.

At his stage of our course we consider only first order equations.

(1) Equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$. This type should need no explanation and an immediate integration produces the solution $y = \mathrm{F}(x) + A$

(2) Equations which may be rearranged in the form $\mathrm{g}(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$. The left hand side is the derivative of a function of y with respect to x (check the differentiation of an implicit function), hence, integrating with respect to x we have $\int \mathrm{g}(y)\mathrm{d}y = x + A$

(3) The above is a special case of a more general type of equation, one which can be rearranged into the form $\mathrm{g}(y)\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$ which integrates to give $\int \mathrm{g}(y)\mathrm{d}y = \int \mathrm{f}(x)\mathrm{d}x + A$

A convenient aid to memory is to treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ as if it were a fraction and consider the equation to be $\mathrm{g}(y)\mathrm{d}y = \mathrm{f}(x)\mathrm{d}x$.

Equations of this type are said to be 'variable separable'

Ex. Solve the differential equations (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{1-x}$ (b) $\cos^2 x\frac{\mathrm{d}y}{\mathrm{d}x} = \sec y$

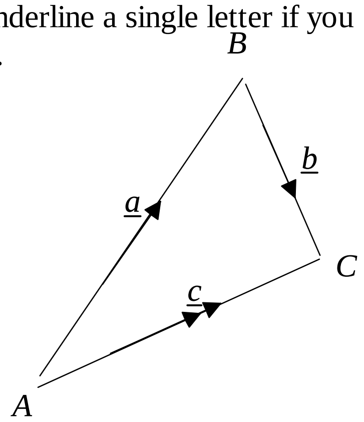
(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{1-x} \Rightarrow \int \frac{1}{y^2}\mathrm{d}y = \int \frac{1}{1-x}\mathrm{d}x \Rightarrow -\int \frac{1}{y^2}\mathrm{d}y = \int \frac{1}{x-1}\mathrm{d}x \int \Rightarrow \frac{1}{y} = \ln A|x-1|$

$$\Rightarrow y = \frac{1}{\ln A|x-1|} = -\ln A|x-1|$$

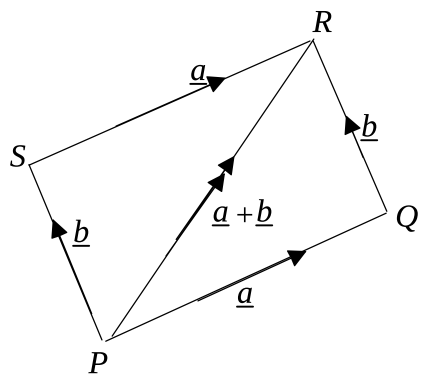
(b) $\cos^2 x\frac{\mathrm{d}y}{\mathrm{d}x} = \sec y \Rightarrow \int \cos y\mathrm{d}y = \int \sec^2 x\mathrm{d}x \Rightarrow \sin y = \tan x + A$

Vectors

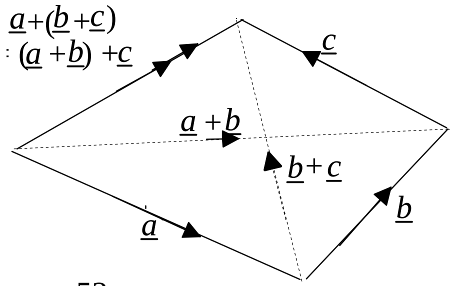
Many physical quantities (eg temperature, pressure, mass, electric charge etc) are completely described by a single measure, the number of the appropriate units for the quantity being described. Such quantities are called SCALAR QUANTITIES or simply SCALARS.
There are however, other quantities, such as the displacement, velocity or acceleration of an object, the intensity of a magnetic fields, forces etc. Which are not completely described by a magnitude but also have an asssociated direction to be specified. These are known as VECTOR QUANTITIES or VECTORS.
In diagrams it is common practice to represent a vector by a straight line with its length representing the magnitude of the vector and its direction relative to some fixed line representing the direction of the vector. An arrow head is usually placed on the line to indicate the sense of the direction along the line.
There are two common methods of describing a vector, (i) in polar, or magnitude direction form (r, θ) where r is the magnitude and θ the direction measured clockwise from the positive x – axis.
(ii) in component form (x, y) which may also be written $x\mathbf{i}+y\mathbf{j}$ where **i** and **j** are lines of unit length in Two given perpendicular directions, usually, **i** along the x-axis and **j** along the y-axis. Another notation for this is $\begin{pmatrix} x \\ y \end{pmatrix}$ In printed work, a vector may be represented by a line from P to Q say, when it is referred to as **PQ** or $\overline{PQ}$ or $\overrightarrow{PQ}$ or by placing a small letter, eg a against the arrowhead, when we refer to the vector **a** or $\underline{a}$. Note that you MUST either draw a line or arrow above the two letters or underline a single letter if you want it to represent a vector, otherwise it is only taken to mean the magnitude.



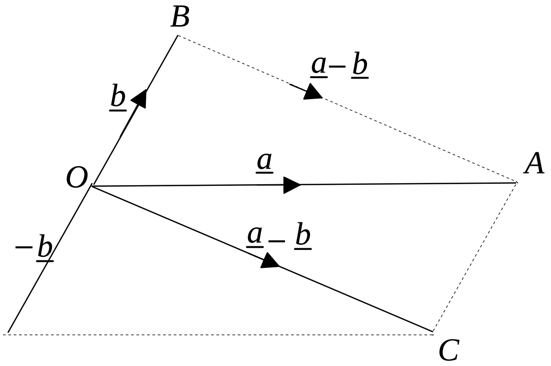
Consider two straight line movements from A to B and then from B tro C
Clearly the effect is the same if we go direct from A to C
and we write $\overline{AB} + \overline{BC} = \overline{AC}$ or $\underline{a} + \underline{b} = \underline{c}$
This is the vector law of addition and we say that $\underline{c}$ is the vector sum of $\underline{a}$ and $\underline{b}$
Displacements such as these are examples of 'localised' or 'fixed' vectors since they are fixed in space. Often however, we wish to concentrate only on the magnitude and direction of a vector quantity and not be concerned with the starting and finishing points.
When we do this, considering the vectors as being 'dislocated' from any particular position, we call them 'free' vectors. Such free vectors are usually denoted by underlined single lower case letters, $\underline{a}$, $\underline{b}$, $\underline{c}$ etc placed by the line representing the vector in a diagram. Note that in printed text in books and examination papers it is common practice to use bold face type instead of underlining.
One approach to building an 'algebra of vectors' is the abstract one of defining a free vector geometrically as the set of all directed line segments having the same magnitude, direction and sense, and to consider any particular vector to be simply a representative of that set. We can then define vector addition as follows:



Suppose **a** and **b** are two given vectors. Choose any representative, say **PQ** of a and the the particular representative of **b** which starts at the point Q, say **QR**. Then **PR** is a representative of **a + b**. From the diagram it should be clear that we could equally well have taken **SR** and **PS** as representatives of **a** and **b** showing that **a + b = b + a** ie addition is commutative. We can also see now why the addition law is often called the parallelogram law of vector addition
The next diagram illustrates the associativity of vector addition
Thus we may write simply **a + b + c**



A vector with the same magnitude as **a** but exactly the opposite direction is conveniently represented as **– a** for we then have **a + (–)a = OA + AO = OO = 0**, the zero vector, which is the only vector without an associated direction.



We can now define subtraction of vectors as the addition of the additive inverse, thus **a – b = a + (– b)**

The diagram shows clearly that if **OA = a** and **OB = b** then **a – b** may be represented by either **OC** or **BA**.

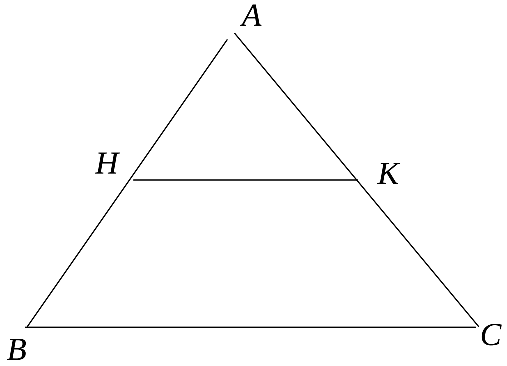
The latter representation is particularly useful and we have **BA = a – b** where **a** and **b** are the POSITION VECTORS of the points *A* and *B* with respect to the *O* as origin.

If λ is a scalar, then $\lambda\mathbf{a}$ is the vector with the same direction as **a** and λ times its magnitude. The following results are reasonably obvious:

If λ, μ are scalars then (i) $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$ (ii) $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$ (iii) $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$

Applications in Geometry

Ex. In a triangle *ABC*, *H* and *K* are the midpoints of *AB* and *AC* respectively. Prove that



(i) *HK* is parallel to *BC*

(ii) $HK = \frac{1}{2}BC$

Let **AB = b** and **AC = c**

Then $\mathbf{AH} = \frac{1}{2}\mathbf{b}$, $\mathbf{AK} = \frac{1}{2}\mathbf{c}$ and $\mathbf{BC} = \mathbf{c} - \mathbf{b}$

So $\mathbf{HK} = \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{c} - \mathbf{b}) = \frac{1}{2}\mathbf{BC}$

Which proves both parts of the question.

This is the mid-point theorem of elementary geometry.

Components

By multiplying by suitable scalars and adding or subtracting, it should be clear that any point in two-dimensional space can be reached from a chosen origin by appropriate combinations of any two non-parallel vectors.

Suppose we choose vectors **a** and **b** as shown, then any other point *P* in the plane can be represented by the vector OP= $\lambda\mathbf{a} + \mu\mathbf{b}$for some values of λ and μ

We say that the vectors **a** and **b** form a BASIS for the two dimensional space.

Also, this representation is unique, for if we also have OP = $\lambda'\mathbf{a} + \mu'\mathbf{b}$ then $\lambda'\mathbf{a} + \mu'\mathbf{b} = \lambda\mathbf{a} + \mu\mathbf{b}$
$\Rightarrow (\lambda' - \lambda)\mathbf{a} = (\mu' - \mu)\mathbf{b}$ and since **a** and **b** are not parallel,

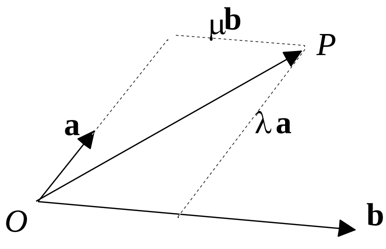
this statement can only be true if $\lambda' - \lambda = \mu' - \mu = 0$

ie $\lambda' = \lambda$ and $\mu' = \mu$

Similarly, the position vector of any point in three-dimensional space, with respect to a given origin, may be uniquely expressed in the form

$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ where, **a**,**b** and **c** are any three non-parallel , non-coplanar Vectors.

λ, μ and ν are the COMPONENTS of the vector for the gi



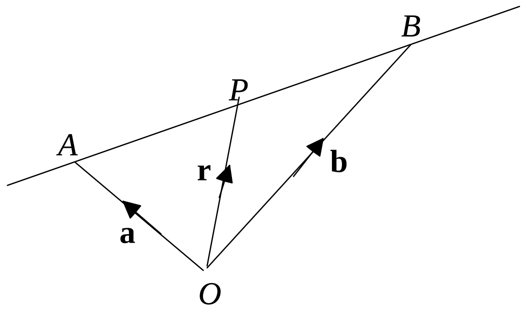
Base Vectors

The most common basis for a component representation of three-dimensional space is the set a three mutually perpendicular vectors of unit magnitude denoted by **i**,**j** and **k** and chosen to form a right-handed system (ie a rotation from **i** to **j** moves a right handed screw in the direction of **k**) In two dimensions it is usual to just use **i** and **j**. Vectors of unit magnitude are known as unit vectors and, apart from **i**,**j** and **k** they are denoted by placing ∧ above the letter. eg **â** is the unit vector in the direction of **a**.

Since the magnitude of **a** is commonly denoted by |**a**| or just a, we have $\mathbf{a} = a\hat{\mathbf{a}}$ or $\mathbf{a} = |a|\hat{\mathbf{a}}$. The magnitude of **AB** is written as |**AB**| or AB.
The rules of vector algebra can now be stated in component form.
(i) Equality: $x_1\mathbf{i} + y_2\mathbf{j} + z_3\mathbf{k} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k} \Leftrightarrow x_1 = x_2,\ y_1 = y_2$ and $z_1 = z_2$.
(ii) Addition/subtraction: $x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \pm (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) = (x_1 \pm x_2)\mathbf{i} + (y_1 \pm y_2)\mathbf{j} + (z_1 \pm z_2)\mathbf{k}$
(iii) Multiplication by a scalar: $\lambda(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \lambda x\mathbf{i} + \lambda y\mathbf{j} + \lambda z\mathbf{k}$.

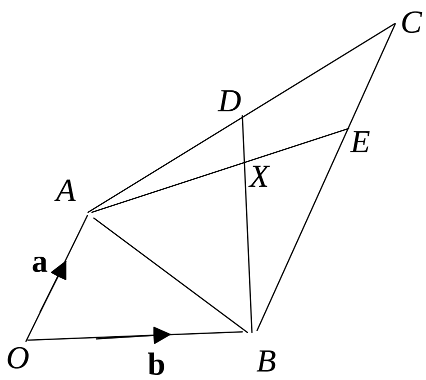
Vector equation of a straight line.

(i) Let a,b be the position vectors of points A,B on a given line. Then if P is any other point on the line, with position vector r we must have
AP = λAB for some real value of $\lambda \Rightarrow$ r – a = λ(b – a)
which is therefore the equation of the line AB.



(ii) Since **b** – **a** is simply a vector in the direction of AB, it follows that if a is the position vector of any point on a line and **c** is any vector parallel to the line then the equation of the line may be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{c}$.
Ex. A and B have position vectors **a** and **b** with respect to some origin O. The position vector of C is 2**a** + **b**. D is the mid-point of AC and E is a point on BC such that $BE = 2EC$. If AE and BD meet at X find the position vector of X in terms of **a** and **b**.

D has position vector $\frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(3\mathbf{a} + \mathbf{b})$
so $\mathbf{BD} = \frac{1}{2}(3\mathbf{a} + \mathbf{b}) - \mathbf{b} = \frac{1}{2}(3\mathbf{a} - \mathbf{b})$
The equation of BD is thus $\mathbf{r} = \mathbf{b} + \lambda(3\mathbf{a} - \mathbf{b})$
Note that we can ignore the $\frac{1}{2}$ since it may be considered to be included in the constant λ
Similarly, E has position vector $\mathbf{b} + \frac{2}{3}(\mathbf{c} - \mathbf{b})$
ie $\mathbf{b} + \frac{2}{3}(2\mathbf{a}) = \frac{1}{3}(4\mathbf{a} + 3\mathbf{b})$
So $\mathbf{AE} = \frac{1}{3}(4\mathbf{a} + 3\mathbf{b}) - \mathbf{a} = \frac{1}{3}(\mathbf{a} + 3\mathbf{b})$ and the equation of AE is $\mathbf{r} = \mathbf{a} + \mu(\mathbf{a} + 3\mathbf{b})$



So at X we must have $\mathbf{a} + \mu(\mathbf{a} + 3\mathbf{b}) = \mathbf{b} + \lambda(3\mathbf{a} - \mathbf{b})$
Equating **a** and **b** components we have
$1 + \mu = 3\lambda$ and $3\mu = 1 - \lambda \Rightarrow \lambda = \frac{2}{5}$ and $\mu = \frac{1}{5}$
The position vector of X is thus $\mathbf{a} + \frac{1}{5}(\mathbf{a} + 3\mathbf{b}) = \frac{1}{5}(6\mathbf{a} + 3\mathbf{b})$

Cartesian equations of a line in 3-dimensional space.

From the vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where $\mathrm{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ we have $x = a_1 + \lambda b_1,\ y = a_2 + \lambda b_2$ and $z = a_3 + \lambda b_3$ or $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} (= \lambda)$These are the cartesian equations of the line.
Ex. Find the cartesian equations of the line through the point (2,3,0) parallel to $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
We have $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \Rightarrow x = 2 + \lambda,\ y = 3 - \lambda$ and $z = 2\lambda \Rightarrow x - 2 = 3 - y = \frac{z}{2}$Ex. Find a vector equation for the line with cartesian equations $\frac{x+4}{3} = \frac{y-5}{2} = \frac{2-z}{4}$
The line passes through the point (–4, 5, 2) and is parallel to $3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ so the vector equation is
$\mathbf{r} = -4\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

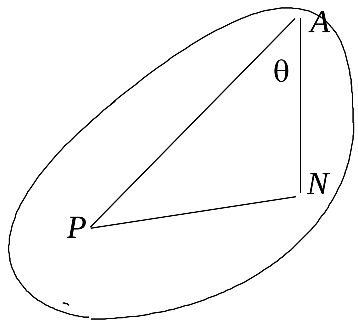
The Scalar (Dot or Inner) Product

We define the scalar product of two vectors **a** and **b**, denoted by **a.b** by **a.b**= |**a**||**b**| $\cos\theta$ where θ is the angle between **a** and **b**. Note that the result is NOT a vector. Since $\cos(-\theta) = \cos\theta$, it doesn't matter whether θ is measured from **a** to **b** or from **b** to **a** but it is usual to take it as the angle between 0 and 180
The following results are consequences of the definition.
(i) **a.a**= $\mathbf{a}^2 = aa\cos 0^o = a^2$
(ii) **a.b** = **b.a**
(iii) $\mathbf{a.b} = 0 \Rightarrow \mathbf{a} = 0$, $\mathbf{b} = 0$ or $\theta = 90^o$, thus $\mathbf{a.b} = \mathbf{0}$ with $\mathbf{a},\mathbf{b} \neq \mathbf{0} \Rightarrow \mathbf{a}$ and **b** are perpendicular.

(iv) **a**.(a**b**) = a(**a**.**b**) = (a**a**).**b** for any scalar a
(v) (**a** + **b**).**c** = **a**.**c** + **b**.**c**
From (i) we have **i**.**i** = **j**.**j** = **k**.**k** =1 whilst from (ii) we have **i**.**j** = **j**.**k** = **k**.**i** = 0 hence, if
$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then $\mathbf{a}.\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$
i.e. $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
An immediate application of this result is to enable us to find the angle between two vectors.
Ex. Find the angle between the vectors 3**i** +2**j** + **k** and –**i** +4**j** –2**k**
We have $(3\mathbf{i} + 2\mathbf{j} + \mathbf{k}).(-\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = -3 + 8 - 2 = 3$
$|3\mathbf{i} + 2\mathbf{j} + \mathbf{k}| = \sqrt{9+4+1} = \sqrt{14}$ and $|-\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = \sqrt{1+16+4} = \sqrt{21}$
hence, $\sqrt{14}\,\sqrt{21}\cos\theta = 3 \Rightarrow \cos\theta = \dfrac{3}{\sqrt{14}\,\sqrt{21}} = 0.1750 \Rightarrow \theta = 79.9^{\circ}$
Ex. Find a unit vector perpendicular to both **i** + 3**j** – 2**k** and 3**i** – 2**j** + 4**k**
We first seek any vector perpendicular to both. Consider the vector a**i** + b**j** + c**k**. This is perpendicular to both the given vectors if $a + 3b - 2c = 0$ and $3a - 2b + 4c = 0$ (scalar products)
Since we only have two equations with three unknowns there is no unique solution but as any solution will do we may take an arbitrary value for one of the unknowns, say take $a = 1$
We then have $3b - 2c = -1$ and $2b - 4c = 3$ so solving simultaneously $b = -\frac{5}{4}$ and $c = -\frac{11}{8}$
The only time this method will fail is if a should have been 0, but in that case we would have obtained a pair of inconsidtent equations in b and c.
Hence, $\mathbf{i} - \frac{5}{4}\mathbf{j} - \frac{11}{8}\mathbf{k}$ is a suitable vector and so also is $8\mathbf{i} - 10\mathrm{j} - 11\mathbf{k}$ with magnitude $\sqrt{64 + 100 + 121}$
ie $\sqrt{285}$ and so $\dfrac{1}{\sqrt{285}}(8\mathbf{i} - 10\mathbf{j} - 11\mathbf{k})$ is the required unit vector.

Vector equation of a plane

Just as any straight line in a plane has an equation of the form $ax + by + c = 0$ so any plane in three-dimensional space has an equation of the form $ax + by + cz = \text{constant}$, with respect to mutually perpendicular axes Ox, Oy, Oz.
Let
P be the point with position vector **r** = x**i** + y**j** + z**k**, and let **n** be tl
hen this equation may be written as .**r**.**n** = constant. If A is any point in the plane with position vector a, then putting **r** = **a**, the equation may be written **r**.**n** = **a**.**n** ⇒ (**r** – **a**).**n** = 0



Now **r – a** = **AP**, a vector in the plane, hence, **n** must be perpendicular to the plane. Thus the equation of any plane may be written in the form **r**.**n** = constant, where n is any vector perpendicular to the plane.
If **n** is a unit vector perpendicular to the plane then we write **r**.**n** = p and we can see that p is the perpendicular distance from the origin to the plane. It is usual to specify **n** so that p is positive.
A consequence of the above reasoning is that we can easily write down a vector perpendicular to a plane from its cartesian equation. If the equation is $ax + by + cz = k$ then a**i** +b**j** +c**k** is perpendicular to the plane. Conversely, if a**i** +b**j** +c**k** is perpendicular to a plane then the equation of the plane must be $ax + by + cz = \text{constant}$ and the constant is easily found if we know any one point of the plane. Taking A as origin, let **AP** = **r**, and let **n** be a unit vector
in the direction of **AN**, then clearly $AN = AP\cos\theta$
ie $AN = \mathbf{r}.\mathbf{n}$

Applications

Ex. Find the distance of the points $A(2, 1, 0)$ and $B(-3, 2, -1)$ from the plane $x - y + 2z = 3$.
Let N be the foot of the perpendicular from A to the plane and let P be any other point in the plane. Then $AN = \dfrac{\mathbf{AP}.\mathbf{AN}}{|\mathbf{AN}|} = \mathbf{AP}.\mathbf{n}$ where n is a unit normal to the plane. Now **i – j** + 2**k** is a normal to the plane and we may take P to be (1,1,0) so **AP** = – **i**
Distance of A from the plane is thus $\dfrac{(-\mathbf{i}).(\mathbf{i} - \mathbf{j} + 2\mathbf{k})}{\sqrt{6}} = -\dfrac{1}{\sqrt{6}}$
Similarly **BP** = 4**i** – **j** + **k** so distance of B from plane is $\dfrac{(4\mathbf{i} - \mathbf{j} + \mathbf{k}).(\mathbf{i} - \mathbf{j} + 2\mathbf{k})}{\sqrt{6}} = \dfrac{7}{\sqrt{6}}$

The different signs tell us that A and B are on opposite sides of the plane.
Ex. Find the angle betwen the planes $3x - 2y - z = 4$ and $x + 3y - 2z = 7$
The angle between the planes is the same as the angle between the normals to the planes, ie the angle between 3**i** – 2**j** – **k** and **i** + 3**j** – 2**k** which is $\arccos\left|\frac{(3\mathbf{i}-2\mathbf{j}-\mathbf{k}).(\mathbf{i}+3\mathbf{j}-2\mathrm{k})}{\sqrt{9+4+1}\,\sqrt{1+9+4}}\right| = \arccos\frac{1}{14} = 85.9^{\mathrm{o}}$

Ex. Find the position vector of the foot of the perpendicular from the point A (2,3,–1) to the plane $2x - y + z = 12$
Direction of the perpendicular is the direction of the normal to the plane, ie 2**i** – **j** + **k** so the equation of the perpendicular is **r** = (2**i** + 3**j** – **k**) + λ(2**i** – **j** + **k**) and we simply have to solve this equation and that of the plane simultaneously. From the equation of the line we have
$x = 2 + 2\lambda,\ y = 3 - \lambda$ and $z = -1 + \lambda$ so substituting in equation of plane gives
$4 + 4\lambda - 3 + \lambda - 1 + \lambda = 12 \Rightarrow 6\lambda = 12$ so $\lambda = 2$ and point is $(6, 1, 1)$