Further Concepts for Advanced Mathematics FP1

Proof

There are three main techniques of proof used in Mathematics

(i) A logical sequence of steps leading from some known fact to conclude with what we wish to prove.

(ii) Proof by contradiction. ie, showing that the assumption that a statement is false leads to a mathematical contradiction.

(iii) Proof by induction .

The first two cases were dealt with earlier in the course.

Proof by induction

Suppose that $\mathrm{P}(n)$ is a statement concerning the natural number n, i.e. $\mathrm{P}(n)$ is a function whose domain is $\mathbb{N}$, the set of natural numbers.

The PRINCIPLE OF MATHEMATICAL INDUCTION states that if

(i) $\mathrm{P}(n_0)$ is true for some $n_0 \in \mathbb{N}$ and (ii) $\mathrm{P}(k) \Rightarrow \mathrm{P}(k+1)$ for any $k \geq n_0$, then $\mathrm{P}(n)$ is true for all $n \geq n_0$

Proof: If the above is not true then there exists a subset of the natural numbers consisting of those numbers greater than n_0 for which $\mathrm{P}(n)$ is false. By the well-ordering property of the natural numbers , This subset has least member m say. i.e. $m > n_0$ and $\mathrm{P}(m)$ is false. However, $\mathrm{P}(m-1)$ must be true, since m is the least member of the subset and then, by the principle of induction $\mathrm{P}(m)$ is true contradicting the assumption that it is false. Hence, no such subset can exist.

Ex. Prove that for all $n \in \mathbb{N}$, $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Let $\mathrm{P}(n)$ be the statement that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

When $n = 1$, $\sum_{r=1}^{n} r = 1$ and $\frac{1}{2}n(n+1) = \frac{1}{2} \times 1 \times 2 = 1$ so $\mathrm{P}(1)$ is true.

Assume $\mathrm{P}(k)$ true, i.e. $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$ then $\sum_{r=1}^{k+1} r = \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2)$

But this is $\mathrm{P}(k+1)$ so $\mathrm{P}(k) \Rightarrow \mathrm{P}(k+1)$ and hence $\mathrm{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n

Another important use for the induction method of proof is to test divisibility statements.

Ex. Prove that $13^n - 6^{n-2}$ is divisible by 7 for all positive integers greater than m, wher m is to be found.

Let $\mathrm{f}(n) = 13^n - 6^{n-2}$ then $\mathrm{f}(1) = 13 - 6^{-1}$ which is not an integer.

$\mathrm{f}(2) = 13^2 - 6^0 = 169 - 1 = 168 = 7 \times 26$ so $\mathrm{f}(2)$ is divisible by 7 so we take $m = 2$.

Assume now that $\mathrm{f}(k) = 13^k - 6^{k-2} = 7X$ where X is an integer, for some $k \geq 2$

Now $\mathrm{f}(k+1) = 13^{k+1} - 6^{k-1} = 13(13^k - 6^{k-2}) + 13 \times 6^{k-2} - 6^{k-1}$

$$= 13 \times 7X + 6^{k-2}(13-6) = 7(13k + 6^{k-2})$$

And since $13k + 6^{k-2}$ is an integer for $k \geq 2$ it follows that $\mathrm{f}(k+1)$ is divisible by 7.

Hence, true for $n = k \Rightarrow$ true for $n = k+1$ and the result follow for all $n \geq 2$ by induction on n.

Sequences and Series

A function whose domain is the set of natural numbers is called a SEQUENCE. The elements of the range of the function are the TERMS of the sequence. There are two common methods of defining a sequence

(1) By giving a formula for the general term. Which again is done in one of two ways.

Ex. $f(n) = 2n - 1$ defines the sequence 1, 3, 5, 7, ...

Ex. $\{n^3 - 1\}$ defines the sequence 0, 7, 26, 63, ...

(2) By expressing the n^{th} or $(n+1)^{\text{th}}$ term in terms of the previous term or sometimes the previous two terms. u_n is commonly used to denote the n^{th} term of a sequence.

Ex. $u_{n+1} = nun$ with $u_1 = 1$ defines the sequence 1, 1, 2, 6, 24, ...

Ex. $u_{n+1} = 3u_n - u_{n-1}$ defines the sequence 1, 1, 2, 5, 13, ...

These are called RECURRENCE RELATIONS

Series

The sum of the first n terms of a sequence forms a FINITE SERIES. If all the terms of a sequence are added we have an INFINITE SERIES

e.g. $1+2+4+8+16+32$ is a finite series of six terms.

$1+2+4+8+16+32+\ldots+2^n+\ldots$ is an INFINITE SERIES

Behaviour of series

Sequences and series may behave in a number of ways as the number of terms increase. The sum of the series may

(i) Tend to some finite limiting value. We say that it CONVERGES to that value

(ii) Increase in size without limit. We say it DIVERGES

(iii) OSCILLATE between finite or infinite limits.

(iv) Display PERIODICITY. i.e the same sequence of values repeat at regular intervals.

Ex. (i) $1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n+1}}+\ldots$ converges to the value 2. We say the sum to infinity is 2

(ii) $1+2+4+8+16+\ldots+2^n+\ldots$ diverges. The sum tends to infinity.

(iii) $1-2+4-8+16-\ldots$ oscillates

(iv) $\{1-(-1)^n\} = \{2, 0, 2, 0, \ldots\}$ displays periodicity

Note that a sequence is a collection of numbers whereas a finite series has a unique finite numerical value. An infinite series may or may not have a finite value.

We may occasionally relax the requirement that the domain of our function be the set of natural numbers and allow integer values or omit the first few values. We commonly use the sigma notation when dealing with series. i.e. $\sum_{r=1}^{n} f(r)$ which means that we add together the values of $f(r)$ for all integer values of r between 1 and n inclusive.

Ex. (i) $\sum_{r=1}^{r=4} r = 1+2+3+4 = 10$ (ii) $\sum_{1}^{5}(3r-1) = 2+5+8+11+14 = 40$

Note also that we often omit the $r =$ when it can cause no confusion.

(iii) $\sum_{3}^{6}(r^2-5) = 4+111+20+31 = 66$

Note also that the same series may be defined in many different ways

Ex. $\sum_{1}^{n} r(r-1) = \sum_{0}^{n-1} r(r+1) = \sum_{2}^{n+1}(r-1)(r-2)$ etc.

Summation of Finite Series

We have previously met three special series, the arithmetic, geometric and binomial series.
We now consider some special techniques.

Sums of Powers of Natural Numbers

We already know that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

The first result is immediate from the sum of an arithmetic series whilst the second may be obtained as follows: $(r+1)^3 - r^3 = 3r^2 + 3r + 1 \Rightarrow r^2 = \frac{1}{3}\left[(r+1)^3 - r^3 - 3r - 1\right]$

$$\Rightarrow \sum_{r=1}^{n} r^3 = \frac{1}{3}\left(\sum_{r=1}^{n}\left[(r+1)^3 - r^3\right] - 3\sum_{r=1}^{n} r - n\right) = \frac{1}{3}\left((n+1)^3 - 1 - \frac{3}{2}n(n+1) - n\right)$$
$$= \frac{1}{3}n(n+1)[2(n+1)^2 - 3n - 2] = \frac{1}{6}(n+1)(2n^2+n) = \frac{1}{6}n(n+1)(2n+1)$$

Similarly, starting with the identity $(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$ we can deduce the sum of the cubes of the natural numbers. This is left as an exercise.

These results can be use to sum other series.

Ex Find the sum of n terms of $1\times 1 + 2\times 3 + 3\times 5 + 4\times 7 + \ldots + n(2n-1)$

As the r^{th} term is $r(2r-1)$ we have $S_n = \sum_{r=1}^{n} r(2r-1) = 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$

$$= \frac{1}{3}n(n+1)(2n+1) - \frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(2(2n+1)-3) = \frac{1}{6}n(n+1)(4n-1)$$

The Method of Differences.

If, for each r, we can write the r^{th} term of a series in the form $\mathrm{f}(r+1) - \mathrm{f}(r)$ then $S_n = \mathrm{f}(n+1) - \mathrm{f}(1)$

Ex Prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

As an alternative to using the sum of an arithmetic series, note that $r = \frac{1}{2}[r(r+1) - (r-1)r]$

i.e. $r = \mathrm{f}(r+1) - \mathrm{f}(r)$ where $\mathrm{f}(r) = \frac{1}{2}(r-1)r$, hence, $\sum_{r=1}^{n} r = \mathrm{f}(n+1) - \mathrm{f}(1) = \frac{1}{2}(n+1)n$ since $\mathrm{f}(1) = 0$

Ex Find the sum of $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{n(n+1)}$

By partial fractions $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} = \mathrm{f}(n) - \mathrm{f}(n+1) = -[\mathrm{f}(n+1) - \mathrm{f}(n)]$ where $\mathrm{f}(n) = \frac{1}{n}$

hence, $\sum_{r=1}^{n} \frac{1}{n(n+1)} = \mathrm{f}(1) - \mathrm{f}(n+1) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

Sometimes a series can be reduced to a standard recognisable form.

Ex Find the sum of $x + 2x^2 + 3x^3 + \ldots + nx^n$

If $S_n = x + 2x^2 + 3x^3 + \ldots + nx^n$ then $xS_n = x^2 + 2x^3 + 3x^4 + \ldots + nx^{n+1}$

so by subtraction $S_n - xS_n = x + x^2 + x^3 + \ldots + x^n - nx^{n+1} = \frac{x(1-x^n)}{1-x} - nx^{n+1}$ (by sum of G.P)

i.e. $S_n = \frac{x[1-x^n-nx^n(1-x)]}{1-x} = \frac{x(1-(1+n)x^n+nx^{n+1})}{1-x}$

Complex Numbers

Though the number system has already been extended in many ways, starting with the natural numbers, through the integers and rational numbers to the real number system, there are still many quite simple equations which have no solutions within this system, e.g. $x^2+2=0,\ \sin x=2,\ 10^x=-5$ etc

Consider quadratic equations, we know that the general quadratic $ax^2+bx+c=0$ has solutions $x=\frac{1}{2a}\left(-b\pm\sqrt{b^2-4ac}\right)$ and that there are no real roots if $b^2-4ac<0$

Suppose now that $b^2-4ac=-k=k(-1)$ where $k>0$, then $\sqrt{b^2-4ac}=\sqrt{k}\sqrt{-1}$ so we may write the solutions as $x=p\pm q\sqrt{-1}$

Let us now introduce a special symbol to denote $\sqrt{-1}$. We shall use j (though i is also often used)

Thus, if the roots of a quadratic equation are not real, we may write $x=p\pm q\mathrm{j}$

We define a COMPLEX NUMBER to be a number of the form $a+b\mathrm{j}$ where a and b are real, the + sign being merely a means of linking the two parts of the number together.

"a" is called the REAL part and "b" the IMAGINARY part of the number, hence if $z=a+b\mathrm{j}$ then $\mathrm{Re}(z)=a$ and $\mathrm{Im}(z)=b$. Note that the imaginary part of a complex number is real, since it is the coefficient of j and does not include j itself.

Manipulation of complex numbers

Operations on complex numbers are defined as follows by assuming the usual properties of numbers to hold together with the assumption that $\mathrm{j}^2=-1$

thus: Equality $a+b\mathrm{j}=c+d\mathrm{j} \Leftrightarrow a=c$ and $b=d$

Addition/subtraction $(a+b\mathrm{j})\pm(c+d\mathrm{j})=(a\pm c)+(b\pm d)\mathrm{j}$

Multiplication $(a+b\mathrm{j})(c+d\mathrm{j})=ac+(bc+ad)\mathrm{j}-bd=(ac-bd)+(bc+ad)\mathrm{j}$

Division Since $(a+b\mathrm{j})(a-b\mathrm{j})=a^2+b^2$ we have $\frac{a+b\mathrm{j}}{c+d\mathrm{j}}=\frac{a+b\mathrm{j}}{c+d\mathrm{j}}\times\frac{c-d\mathrm{j}}{c-d\mathrm{j}}=\frac{(ac+bd)+(bc-ad)\mathrm{j}}{c^2+d^2}$

from these definitions we can see that the complex numbers of the form $a+0\mathrm{j}$ behave in exactly the same way as the real numbers and it is usual to omit the 0j. Similarly, purely imaginary numbers $0+b\mathrm{j}$ are usually written simply as $b\mathrm{j}$.

If $z=a+b\mathrm{j}$, then the multiplicative inverse z^{-1} of z is given by $z^{-1}=\frac{1}{a+b\mathrm{j}}=\frac{a-b\mathrm{j}}{a^2+b^2}$ providing $z\neq 0$

The number $z^*=a-b\mathrm{j}$ is called the COMPLEX CONJUGATE (or simply the conjugate) of z

The real number $|z|=\sqrt{a^2+b^2}$ is called the MODULUS of z, thus $z^{-1}=\frac{z^*}{|z|^2}$ or $|z|^2=\frac{z^*}{z^{-1}}=zz^*$

Some important results concerning complex conjugates are:

If $z_1=x_1+y_1\mathrm{j}$ and $z_2=x_2+y_2\mathrm{j}$ with $z_2\neq 0$ then

$(z_1+z_2)^*=z_1^*+z_2^*;\ (z_1z_2)^*=z_1^*z_2^*;\ \left(\frac{z_1}{z_2}\right)^*=\frac{z_1^*}{z_2^*};\ z_1+z_1^*=2\mathrm{Re}(z_1)$ and $z_1-z_1^*=2\mathrm{j}\,\mathrm{Im}(z_1)$

these results are most important and should be memorised

The square root of a complex number

The method is explained by an example.

Ex Find $\sqrt{21-20\mathrm{j}}$

Let $21-20\mathrm{j}=(a+b\mathrm{j})^2=(a^2-b^2)+2ab\mathrm{j}$ then we must have $a^2-b^2=21$ and $2ab=-20$

It is fairly obvious, by inspection, that the possible solutions are $a=5,\ b=-2$ and $a=-5,\ b=2$ If you cannot "see" these solutions then you must solve $a^2-b^2=21$ and $2ab=-20$ simultaneously.

Thus we have $\sqrt{21-20\mathrm{j}}=5-2\mathrm{j}$ or $-5+2\mathrm{j}$. Note that since the complex numbers do not form an 'ordered' set, we cannot say that one of these roots is positive and the other negative.

Consider the polynomial equation $a_nx^n+{}_{n-1}x^{n-1}+\ldots+a_1x+a_0=0$ with all a_i real. It can be proved that such an equation always has at least one root in Š, the set of complex numbers. Remember that Š contains ‘ . Let such a root be $z=a+b\mathrm{j}$. then $a_nz^n+{}_{n-1}z^{n-1}+\ldots+a_1z+a_0=0$

Taking the complex conjugate of both sides and using the standard results given above we have, since $a_i^*=a_i$ then $a_nz_n^*+a_{n-1}z_{n-1}^*+\ldots+a_1z^*+a_0=0\Rightarrow z^*=a-b\mathrm{j}$ is also a root of the equation

Thus, complex roots of a polynomial equation with real coefficients, always occur in conjugate pairs.

What are the important consequences of this ?

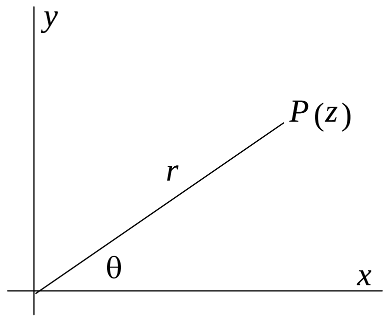
Ex Solve the equation $z^3+z^2-4z+6=0$ given that 1 +j is a root.

If 1 +j is a root then so also is 1 –j hence $z-(1+\mathrm{j})$ and $z-(1-\mathrm{j})$ are factors of the polynomial and hence so is $[z-(1+j)][z-(1-j)]=[(z-1)-j][(z-1)+j]=(z-1)^2+1=z^2-2z+2$

and so $z^3+z^2-4z+6=(z^2-2z+2)(z+3)$ and the solutions are $z=1+\mathrm{j},\ 1-\mathrm{j}$ or -3

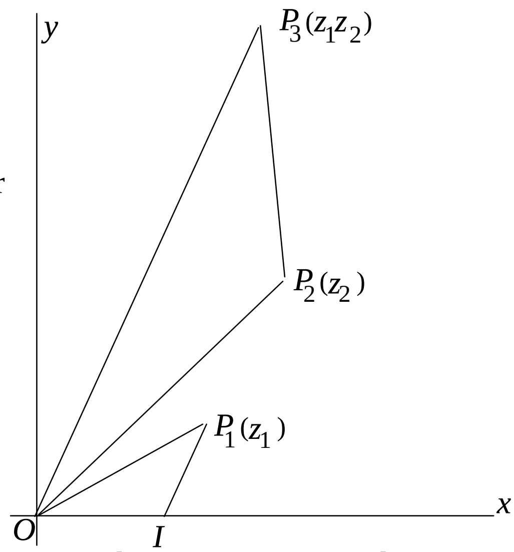
The Argand Diagram

Since a complex number is essentially just an ordered pair of real numbers there is a natural one-to-one correspondence between complex numbers and points in a plane defined by the mapping $x+y\mathrm{j} \leftrightarrow (x,y)$
Points on the x-axis correspond to the real numbers and points on the y-axis to the purely imaginary numbers and so we refer to the real and imaginary axes.
Such a representation of the complex numbers is known as the COMPLEX PLANE or the ARGAND DIAGRAM (J.R.Argand 1768-1822)
The point $P(x,y)$ corresponding to $z=x+y\mathrm{j}$ is called the AFFIX of z
The Argand diagram gives us an alternative interpretation of j , not as a number, but an operator, an instruction to perform a certain geometrical transformation.
Let $z=a+b\mathrm{j}$ be any complex number with affix P. Then $\mathrm{j}z=a\mathrm{j}-b$ with affix P' is the image of P under a positive (anti-clockwise) rotation of 90° about the origin. Thus j can be interpreted as an operator which causes a positive 90° rotation about the origin.
It should be obvious from the definition of addition/subtraction that in the Argand diagram it is exactly the same as adding/subtracting vectors. The MODULUS which has already been defined can now be seen to be the distance of the affix of a number from the origin. Before considering multiplication and division in the Argand diagram we need to introduce the concept of the ARGUMENT of a complex number.
The position vector of the point representing a complex number z in the Argand diagram can be described by means of its length r and the angle θ that it makes with the positive real axis.
r is of course $|z|$ the modulus of z. The angle θ, which is measured anticlockwise from the positive real axis, usually in radians, is not however uniquely defined since adding any multiple of 2π will give the same direction. It is usual therefore to take that value of θ for which $-\pi<\theta\le\pi$, which is called the PRINCIPAL ARGUMENT of z, denoted by $\arg z$



If $\arg(z)=\theta$ we have $z=r\cos\theta+(r\sin\theta)\mathrm{j}$
ie $z=r(\cos\theta+\mathrm{j}\sin\theta)$ where $\tan\theta=\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$
This is the MODULUS-ARGUMENT or POLAR form of z.
Ex. (i) $1+\mathrm{j}=r(\cos\theta+\mathrm{j}\sin\theta) \Rightarrow r=\sqrt{2}$, $\tan\theta=1\Rightarrow\theta=\frac{\pi}{4}$
so $1+\mathrm{j}=\sqrt{2}(\cos\frac{\pi}{4}+\mathrm{j}\sin\frac{\pi}{4})$
(ii) $3-4\mathrm{j}=5(\cos\theta+\mathrm{j}\sin\theta)$ where $\tan\theta=-\frac{4}{3}$ and $-\frac{\pi}{2}<\theta<0$
Be sure to get the correct quadrant for the argument.

Consider now $z_1=r_1(\cos\theta_1+\mathrm{j}\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+\mathrm{j}\sin\theta_2)$
Then $z_1z_2=r_1r_2(\cos\theta_1+\mathrm{j}\sin\theta_1)(\cos\theta_2+\mathrm{j}\sin\theta_2)$
$$=r_1r_2(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+\mathrm{j}\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)$$
$$=r_1r_2[\cos(\theta_1+\theta_2)+\mathrm{j}\sin(\theta_1+\theta_2)]$$
Thus $|z_1z_2|=|z_1||z_2|$ and $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)\pm2\pi$
Similarly, $|\frac{z_1}{z_2}|=\frac{|z_1|}{|z_2|}$ and $\arg(\frac{z_1}{z_2})=\arg(z_1)-\arg(z_2)\pm2\pi$
Note that the $\pm2\pi$ may be necessary in order to give a principal argument.
Geometrically therefore, multiplication by z enlarges by a factor $|z|$ and rotates through an angle $\arg z$
This is called a SPIRAL DILATION.
In the diagram, I, P_1 and P_2 are the affixes of $1, z_1$ and z_2 respectively. If each side of $\triangle OIP_1$ is enlarged by a factor $|z_2|$ and the triangle rotated through an angle $\arg(z_2)$ then I is mapped onto P_2. Let P_3 be the image of P_1 then $\triangle' s\ OIP_1$ and OP_2P_3 are similar and thus
$|OP_3|=|z_2||OP_1|=|z_2||z_1|$ and $I\hat{O}P_{3=\arg(z_1)+\arg(z_2)}$
and hence, P_3 is the affix of z_1z_2

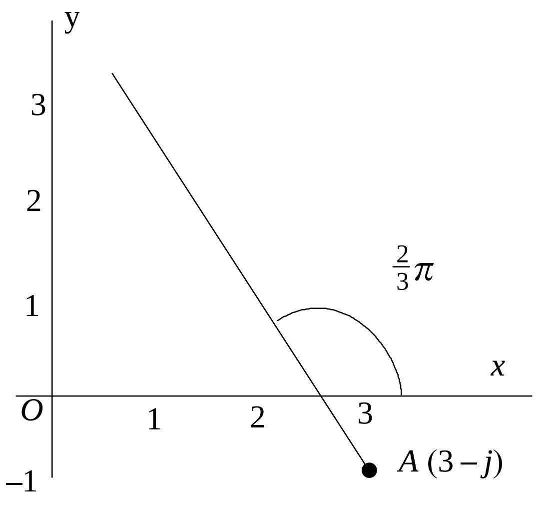


Thus, to construct z_1z_2 we draw on OP_2, a triangle similar to OIP_1, making OP_2 correspond to OI

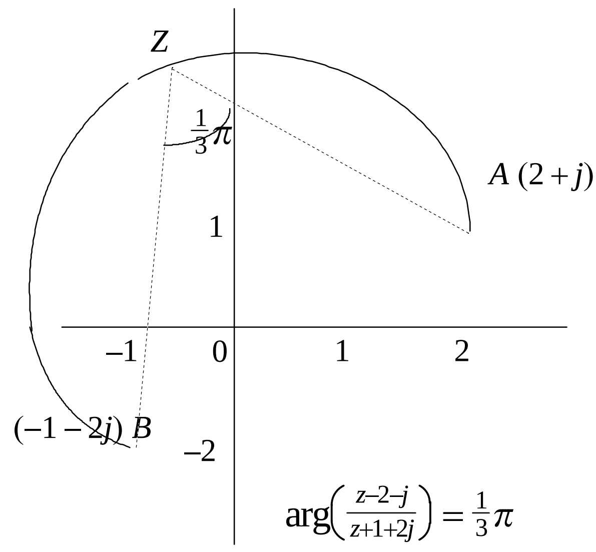
Simple loci in the Argand diagram

From the connection between complex numbers and vectors we see that $|z_1-z_2|$ is the distance between the points representing the complex numbers z_1 and z_2. Thus if A, B and P represent the complex constants a, b and the complex variable z then

(i) if $|z-a| = k$ where k is real, the locus of P is a circle, centre A and radius k.
(ii) if $|z-a| = |z-b|$ then the locus of P is the perpendicular bisector of AB
(iii) if $|z-a| = k|z-b|$ $k \neq 1$ then the locus of P is a circle.
We prove this last result. Let $z = x + \mathrm{j}y$, $a = a_1 + \mathrm{j}a_2$ and $b = b_1 + \mathrm{j}b_2$
then $|z-a| = k|z-b| \Rightarrow |x - a_1 + \mathrm{j}(y-a_2)| = k|x - b_1 + \mathrm{j}(y-b_2)|$
$\Rightarrow (x-a_1)^2 + (y-a_2)^2 = k^2\left[(x-b_1)^2 + (y-b_2)^2\right]$
$\Rightarrow x^2 + y^2 - 2a_1x - 2a_2y + a_1^2 + a_2^2 = k^2(x^2 + y^2 - 2b_1x - 2b_2y + b_1^2 + b_2^2)$
$\Rightarrow (k^2-1)(x^2+y^2) + 2(a_1 - k^2b_1)x + 2(a_2 - k^2b_2)y + k^2(b_1^2+b_2^2) - (a_1^2+a_2^2) = 0$
and dividing by k^2-1 (since $\neq 1$) gives an equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ which is the equation of a circle.
The centre of the circle is at $(-g, -f)$ i.e. the point representing $\frac{k^2b_1 - a_1}{k^2-1} + \mathrm{j}\frac{k^2b_2 - a_2}{k^2-1} = \frac{k^2b - a}{k^2-1}$ which is the point dividing AB externally in the ratio $k^2 : 1$
(iv) if $\arg(z-a) = \theta$. where θ is real, the locus of P is the half-line from A, (excluding A itself) proceeding in the direction θ, measured from the real axis.
(v) if $\arg\left(\frac{z-a}{z-b}\right) = \psi$, the locus of P is a circular arc from A to B such that $\triangle APB = \psi$, measuredfrom BP to AP.
(iv) and (v) are illustrated in the following diagrams.



$\arg(z - 3 + j) = \frac{2}{3}\pi$



$\arg\left(\frac{z-2-j}{z+1+2j}\right) = \frac{1}{3}\pi$

Matrices

A MATRIX is an ordered rectangular array of numbers. Matrices are classified by the number of rows and columns, the ORDER of the matrix. Thus a matrix with 4 rows and 3 columns is said to be a 4×3 Matrix, read as "four-by-three". We usually use capital letters printed in heavy type in examination papers and most text books, to represent matrices except when the matrix consists of a single column when standard vector notation tends to be used, since a vector may be considered to be an $n\times 1$ matrix. Arbitrary elements of a matrix are denoted by small letters often with one or two suffixes as in the following examples.

Ex. $\mathbf{A}=\begin{pmatrix}1&3&5\\4&-1&2\end{pmatrix}$ $\quad\mathbf{B}=\begin{pmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\end{pmatrix}$ $\quad\mathbf{C}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$

A and **B** are 2×3 matrices and **C** is a 3×3 matrix.

Algebra of matrices.

Let $\mathbf{A}=(a_{ij}),\ \mathbf{B}=(b_{ij}),\ \mathbf{C}=(c_{ij})$ where a_{ij} represents the element in row i and column j of **A**

Equality: $\mathbf{A}=\mathbf{B}$ if, and only if, $a_{ij}=b_{ij}$ for every i, j. Clearly **A** and **B** must have the same order.

Addition/subtraction: Providing **A** and **B** have the same order (are COMPATIBLE) then their sum and difference will exist and $\mathbf{A}\pm\mathbf{B}=\mathbf{C}$ where $c_{ij}=a_{ij}\pm b_{ij}$ for every i, j.

Multiplication by a scalar: $k\,\mathbf{A}=(ka_{ij})$

Multiplication of matrices: If **A** has order $r\times s$ and **B** has order $s\times t$ then **AB** will exist and will have order $r\times t$. We say that **A** and **B** are CONFORMABLE for multiplication. Note that the product **BA** will only exist when $r = t$ and even then **AB** and **BA** will not, in general, be equal. It is important to remember that matrix multiplication is NOT commutative.

When it exists, the product **AB** is defined by **AB** = **C** where $c_{ij}=\sum_{r=1}^{s}a_{ir}b_{rj}$ i.e. The element in the i-th row and j-th column of the product **AB** is the "scalar product" of the i-th row of **A** and the j-th column of **B**. If the product does not exist then the matrices are NON-CONFORMABLE.

A square matrix (a_{ij}) of order n with $a_{ii}=1$ for all i and $a_{ij}=0$ for $i\neq j$ is called the IDENTITY matrix of order n and denoted by **I** (or $\mathbf{I}_n$ if we wish to emphsise the order) . This matrix always commutes with any square matrix of the same order. i.e. **AI** = **IA** = **A**

Any matrix in which all elements are zero is called a ZERO or NULL matrix and denoted by **0**

Note that, unlike in basic algebra, **AB** = **0** does NOT imply that either **A** = **0** or **B** = **0**.

Ex. $\begin{pmatrix}3&2\\0&0\end{pmatrix}\begin{pmatrix}0&0\\5&-6\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=\mathbf{0}$

Inverse of a square matrix: If **A**,**B** are square matrices of order n and if **AB** = **I**, then **B** is the INVERSE of **A** and is more usually denoted by $\mathbf{A}^{-1}$. Note that a matrix always commutes with its inverse, i.e. $\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$

Division : There is no operation of division defined on matrices though a similar effect is obtained by multiplying by the inverse of a matrix. Thus, $\mathbf{AB}=\mathbf{C}\Rightarrow\mathbf{B}=\mathbf{A}^{-1}\mathbf{C}$ providing $\mathbf{A}^{-1}$ exists, but remember That you must be very careful with the order of your multiplications.

Ex. $\mathbf{AB}=\mathbf{C}\Rightarrow\mathbf{B}=\mathbf{A}^{-1}\mathbf{C}$ but we must NOT say $\mathbf{B}=\mathbf{CA}^{-1}$

Inverse of a 2 x 2 matrix: If $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ then $\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc$ is the DETERMINANT of **A** .

Alternative notations for the determinant are det **A** or $|\mathbf{A}|$ Providing $ad-bc\neq 0$ the inverse of A will exist and is given by $\mathbf{A}^{-1}=\frac{1}{|\mathbf{A}|}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$. You should memorise this pattern. If $\mathbf{A}^{-1}$ does not exist, which is when the determinant is zero, then we say that **A** is SINGULAR, otherwise **A** is NON-SINGULAR

Simultaneous equations.

The simple simultaneous equations $7x-2y=11$ and $8x-3y=9$, may be written in matrix form as $\begin{pmatrix}7&-2\\8&-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}11\\9\end{pmatrix}$ and the inverse of $\begin{pmatrix}7&-2\\8&-3\end{pmatrix}$ is $-\frac{1}{5}\begin{pmatrix}-3&2\\-8&7\end{pmatrix}$ hence, pre-multiplying each side of the matrix equation by this inverse we have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -3 & 2 \\ -8 & 7 \end{pmatrix}\begin{pmatrix} 11 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

Hence, $x = 3$ and $y = 5$.

Matrices and transformations

An important application of matrices is to represent mappings or transformations. The standard transformations of a plane into itself which leaves the origin invariant, such as reflections in lines through the origin, rotations about the origin, enlargements with centre at the origin etc, may be represented by 2×2 matrices. The images of points of the plane are then obtained by pre-multiplying the position vector of the point by this transformation matrix.

In particular, if $\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix of a transformation, then

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and similarly } \mathbf{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

Remembering this connection between the columns of the matrix and the images of the points (1,0) and (0,1) is the easiest way of identifying a transformation from its matrix, or of finding the matrix of a given transformation.

If we interchange the rows and columns of a matrix **A** we form a new matrix, called the TRANSPOSE of **A**, denoted by $\mathbf{A}^{\mathrm{T}}$. If $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, we say that **A** is SYMMETRIC, whilst if $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$, then **A** is SKEW-SYMMETRIC and if $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$ then **A** is ORTHOGONAL.

If **A** is a 2×2 matrix, then $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$ is symmetric and $\mathbf{A} - \mathbf{A}^{\mathrm{T}}$ is skew-symmetric. Hence every 2×2 matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

A is a symmetric matrix and **B** is skew-symmetric of the same order. **AB** is skew-symmetric if, and only if, **AB** = **BA**.

A and **B** are symmetric matrices of the same order, then **AB** is symmetric if, and only if, **AB** = **BA**. What can be said about $\mathbf{A}^k$ for any positive integer k?

If **A** and **B** are 2×2 matrices, then det(**AB**) = (det**A**)(det**B**)

Invariant points and lines

Points which are mapped onto themselves by a transformation are called INVARIANT points of the transformation.

INVARIANT lines may be of two types.

(i) POINTWISE INVARIANT lines in which every individual point is mapped onto itself, and

(ii) ordinary invariant lines in which any point of the line is mapped onto a point of the line but not necessarily the same point.

We illustrate by examples

Ex. Find the invariant points under the transformation given by the matrix $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is the position vector of an invariant point then we must have $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow 3x + 4y = x$ and $x + 2y = y \Rightarrow 2x + 4y = 0$ and $x + y = 0 \Rightarrow x = 0,\ y = 0$ so only the origin is invariant.

Ex. Find the invariant lines through the origin of the linear transformation with matrix $\begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}$

Consider the image of a typical point (k, mk) on the line $y = mx$

$$\begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} k \\ mk \end{pmatrix} = \begin{pmatrix} 4k - 2mk \\ 5k - 3mk \end{pmatrix} = \begin{pmatrix} \{4 - 2m\}k \\ \{5 - 3m\}k \end{pmatrix}$$

Hence the image of the line $y = mx$ is the set of points $(\{4 - 2m\}k, \{5 - 3m\}k)$ i.e. the line $y = \frac{5-3m}{4-2m}x$

Thus the line $y = mx$ is invariant if $\frac{5-3m}{4-2m} = m \Rightarrow 2m^2 - 7m + 5 = 0 \Rightarrow (2m - 5)(m - 1) = 0 \Rightarrow m = \frac{5}{2}$ or 1

Hence, the invariant lines through the origin are $y = x$ and $y = \frac{5}{2}x$

Ex. A transformation is defined by the matrix $\begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}$. Find the invariant lines.

Any line has an equation of the form $y = mx + c$ with a general point with position vector $\begin{pmatrix} X \\ mX + c \end{pmatrix}$ say. This is mapped onto $\begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} X \\ mX + c \end{pmatrix} = \begin{pmatrix} 5X - 2mX - 2c \\ 2X \end{pmatrix}$ and we want this point also to be on the line $y = mx + c,$ i.e we require that $2X = m(5X - 2mX - 2c) + c$

$\Rightarrow X(2m^2 - 5m + 2) = c - 2mc$ and if this is to hold for all $X,$ then we must have

$2m^2 - 5m + 2 = c(1 - 2m) = 0 \Rightarrow (2m - 1)(m - 2) = -c(2m - 1) = 0 \Rightarrow m = \frac{1}{2}$

Hence, all lines with a gradient of $\frac{1}{2},$ i.e. equations of the form $2y = x + c$ are invariant.

Relations between the coefficients and roots of an equation.

Let the roots of $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ be $\alpha_1,\ \alpha_2, \ldots,\ \alpha_n$ then $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = a_n(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

Comparing coefficients we have $\sum_{i=1}^{n} \alpha_i = -\frac{a_{n-1}}{a_n}$ $\sum_{i,j=1}^{n} \alpha_i\alpha_j = \frac{a_{n-2}}{a_n}$......$\prod_{i=1}^{n} \alpha_i = (-1)^n \frac{a_0}{a_n}$

where $\sum_{i,j=1}^{n} \alpha_i\alpha_j$ means the sum of all products of the roots taken in pairs and $\prod_{i=1}^{n} \alpha_i$ means the product of all the roots. In particular for

(a) $ax^2+bx+c=0$ we have $\alpha_1+\alpha_2 = -\frac{b}{a}$ and $\alpha_1\alpha_2 = \frac{c}{a}$

(b) $ax^3+bx_2+cx+d=0$ we have $\alpha_1+\alpha_2+\alpha_3 = -\frac{b}{a},\ \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = \frac{c}{a}$ and $\alpha_1\alpha_2\alpha_3 = -\frac{d}{a}$

Ex. If α and β are the roots of $ax^2+bx+c=0$, express $(\alpha-2\beta)(2\alpha-\beta)$ in terms of a, b and c. Deduce the condition that one root of the equation is double the other. Find also the condition that one root is the square of the other.

We have $\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ hence,

$(\alpha-2\beta)(2\alpha-\beta) = 2\alpha^2+2\beta^2-5\alpha\beta = 2(\alpha+\beta)^2-9\alpha\beta = \frac{2b^2}{a^2} - \frac{9c}{a} = \frac{2b^2-9ac}{a^2}$

One root is double the other if either $\alpha = 2\beta$ or $\beta = 2\alpha$ and in either case $(\alpha-2\beta)(2\alpha-\beta) = 0$

hence the condition is that $2b^2 = 9ac$

Similarly, one root is the square of the other if $(\alpha-\beta^2)(\alpha^2-\beta) = 0$

Now $(\alpha-\beta^2)(\alpha^2-\beta) = \alpha^3+\beta^3-\alpha^2\beta^2-\alpha\beta$ and$\alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

hence, $(\alpha-\beta^2)(\alpha^2-\beta) = (-\frac{b}{a})^3 - 3\times\frac{c}{a}\times-\frac{b}{a} - (\frac{c}{a})^2 - \frac{c}{a} = \frac{-b^3+3abc-ac^2-a^2c}{a^3}$ and so the required condition is that $b^3+ac^2+a^2c-3abc=0$

Sums of powers of roots of an equation

In the previous example we required $\alpha^2+\beta^2$ and $\alpha^3+\beta^3$ These could have been found as follows:

Since α and β are the roots of $ax^2+bx+c=0$ we have $a\alpha^2 + \Rightarrow b\alpha + c = 0$ and $a\beta^2+b\beta+c=0$ so by addition $aS_2+bS_1+2c=0$ where S_n denotes $\alpha^n+\beta^n$

Thus, since $S_1 = -\frac{b}{a}$ we have $aS_2 - \frac{b^2}{a} + 2c = 0 \Rightarrow S_2 = \frac{b^2-2ac}{a^2}$

Since neither of the roots is zero, they must also satisfy the equation $ax^3+bx^2+cx=0$, hence, putting $x=\alpha$ and $x=\beta$ and adding as before we have a$S_3+bS_2+cS_1=0$ so $S_3 = \frac{-bS_2-cS_1}{a} = \frac{3abc-b^3}{a^3}$

Ex. Find the sum of the cubes of the roots of the equation $x^3-2x^2+3x+1=0$

We cannot find S_2 by this new method so must proceed as follows:

If root are α, β and γ then $\alpha+\beta+\gamma = 2,\ \alpha\beta+\beta\gamma+\gamma\alpha = 3$ and $\alpha\beta\gamma = -1$

hence, $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = 4-6 = 2$

Now we can use our new method to give $S_3-2S_2+3S_1+3=0 \Rightarrow S_3 = 2S_2-3S_1-3 = -4-6-3 = -13$

Equations having given roots.

Reversing the earlier procedure, if the roots of an equation are $\alpha_1,\ \alpha_2\ldots,\ \alpha_n$ then we must have $(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) = 0 \Rightarrow x^n - (\sum\alpha_i)x^{n-1} + (\sum\alpha_i\alpha_j)x^{n-2} + \ldots + (-1)^n \prod\alpha_i = 0$

A very common situation is where we are required to find the equation whose roots are symmetrical functions of the roots of a given equation. The following examples illustrate the technique:

Ex. If $\alpha,\ \beta,\ \gamma$ are the roots of $x^3-3x+2=0$, find the equations whose roots are

(i) $\frac{2}{\int},\ \frac{2}{\beta}$ and $\frac{2}{\gamma}$ (ii) $\alpha^2,\ \beta^2,\ \gamma^2$ (iii) $\alpha+\beta,\ \beta+\gamma,\ \gamma+\alpha$

(i) writing $y = \frac{2}{x}$ we have $x = \frac{2}{y}$ Substituting for x in the given equation will produce an equation with the desired roots. i.e. $(\frac{2}{y})^3 - 3(\frac{2}{y}) + 2 = 0 \Rightarrow 8-6y^2+2y^3=0$ i.e. equation is $2y^3-6y^2+8=0$

(ii) writing $y=x^2$ we have $x=\sqrt{y}$ so $y\sqrt{y}-3\sqrt{y}+2=0 \Rightarrow y\sqrt{y}-3\sqrt{y} = -2 \Rightarrow y^3-6y^2+9y=4$

(iii) Since $\alpha+\beta+\gamma=0$ we have $\alpha+\beta=-\gamma,\ \beta+\gamma=-\alpha$ and $\gamma+\alpha=-\beta$ so the roots of the required equation are $-\alpha,\ -\beta$ and $-\gamma$ hence, writing $y=-x$ we have $x=-y$ and $-y^3+3y+2=0$

i.e. equation is $y^3 - 3y - 2 = 0$

Curve Sketching

Most of the techniques have been dealt with earlier e.g. Intersections with axes, Stationary points, Transformations, Use of standard graphs etc.

The first extra technique is the finding of asymptotes. These are lines which the curve approaches but never actually reaches and are sometimes loosely referred to as ”tangents at infinity”

We consider here the cases of vertical and horizontal asymptotes.

Vertical asymptotes occur for values of x which cannot exist. These are usually values of x which make the denominator of a rational function zero.

Ex $y = \frac{x}{(x-1)(x+2)}$ has vertical asymptotes at $x = 1$ and $x = -2$

Horizontal asymptotes occur when y tends to some finite value as x tends to $\pm\infty$

Ex $y = \frac{1}{x}$ has a horizontal asymptote at $y = 0$ since as $x \to \pm\infty,\ y \to 0$

Ex $y = \frac{3x+2}{x-1} = \frac{3+\frac{2}{x}}{1-\frac{1}{x}} \to 3$ as $x \to \pm\infty$ since $\frac{2}{x}$ and $\frac{1}{x}$ both $\to 0$ so horizontal asymptote is $y = 3$

The next technique is to investigate the behaviour of the function for large ($+$ or $-$) values of x

Ex. $y = \frac{x^2-3x+7}{x+1} = \frac{x(x+1)-4(x+1)+11}{x+1} = x - 4 + \frac{11}{x+1}$ but $\frac{11}{x+1} \to 0$ for large x so $y \to x - 4$

This is in fact an example of an oblique asymptote.

Note however that it is possible for a curve to cross an asymptote at some finite point.

In particular you should be able to easily sketch a quadratic curve by expressing the funcion in completed square form.

Ex To sketch $y = 3x^2 - 6x + 2$ we note that this may be written as $y = 3(x-1)^2 - 1$

y obviously has a minimum value of -1 when $x = 1$, so $x = 1$ is an axis of symmetry and $(1, -1)$ is the lowest point of the graph.

To summarise, with more complex equations, some or all of the following techniques may be required

1. Inspect for symmetry about the axes.

   An even function is symmetrical about the y-axis

   An odd function has rotational symmetry of order two

   If there are only even powers of y then we have symmetry about the x-axis

2. Check for asymptotes

   If $y = \frac{f(x)}{g(x)}$ then $g(a) = 0 \Rightarrow x = a$ is an asymptote and the curve has a discontinuity at $x = a$

   If $y \to k$ as $x \to \pm\infty$ then $y = k$ is an asymptote

3. Determine intersections with axes and asymptotes

4. Investigate stationary points

5. Determine any restrictions on the values of x and y by solving for y and x respectively

The following example illustrates these techniques

Ex. (i) $y = \frac{x(x+1)}{x-1}$

(1) There is no symmetry about either axis (note that ‘negative’ information can be very useful)

(2) $x = 1$ is clearly an asymptote,. also it is clear that $y \to \pm\infty$ as $x \to \pm\infty$ so there are no asymptotes parallel to the x-axis. It is however possible to investigate for other asymptotes as follows

$y = \frac{x(x+1)}{x-1} = \frac{x^2+x}{x-1} = \frac{(x-1)(x+2)+2}{x-1} = x + 2 + \frac{2}{x-1}$ which tends to $x + 2$ as $x \to \infty$

Hence, $y = x + 2$ is an asymptote.

It is also helpful to see from which side the curve approaches the asymptote. This often identifies turning points without the need for differentiation.

Consider the asymptote $x = 1$: if $x = 1 - h$ where h is small, then $y = \frac{(1-h)(2-h)}{-h}$ which will be a large negative number. i.e. as $x \to 1$ from the left, $y \to -\infty$

Similarly, as $x \to 1$ from the right $y \to +\infty$ Again, considering $y = x + 2$ we can see that for $x > 0,\ y > x + 2$ since the largest neglected term is positive, and so the curve tends to this line from above if $x > 0$ and similarly from below if $x < 0$

(3) $x = 0 \Rightarrow y = 0$ but also $y = 0$ when $x = -1$

You should now be able to draw a very good sketch of the graph. Try for yourself and then compare with the graph below.

(4) It is difficult to differentiate and in view of what we already know it is not necessary.

(5) Again it is not really necessary to look for restrictions but we will do it as an example of the technique.

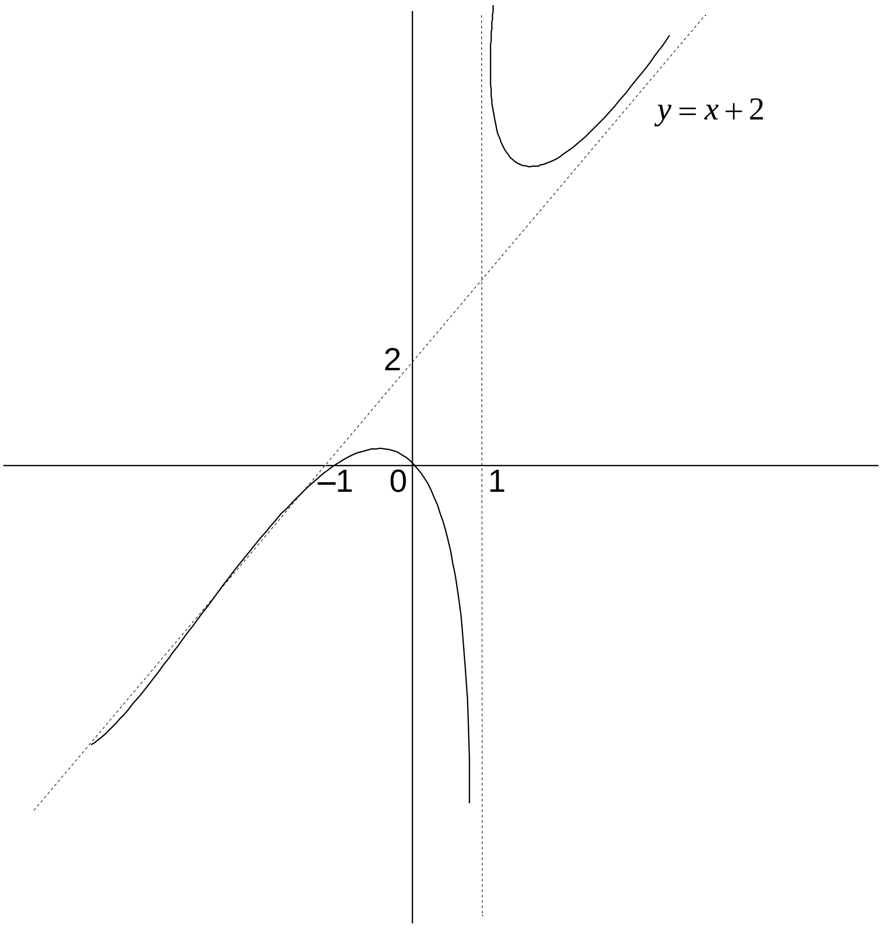
It is obvious that y exists for all values of x except $x = 1$

Solving for x in terms of y we have $xy - y = x^2 + x \Rightarrow x^2 + (1-y)x + y = 0$

$$\Rightarrow x = \frac{y - 1 \pm \sqrt{(1-y)^2 - 4y}}{2} = \tfrac{1}{2}\left[y - 1 \pm \sqrt{1 - 6y + y^2}\,\right]$$

Clearly, the curve can only exist for values of y such that $y^2 - 6y + 1 \geq 0 \Rightarrow (y-3)^2 \geq 8$

i.e. for $y \geq 3 + \sqrt{8}$ or $y \leq 3 - \sqrt{8}$



Polar Coordinates

You are familiar with the representation of a point in a plane by cartesian coordinates (x, y)
The position of a point P is also given by its distance r from the origin and the angle θ between the x axis and the line OP. These are the POLAR COORDINATES of P and are defined with respect to an origin O, called the POLE and a fixed line Ox, called the INITIAL LINE.

The relation between the cartesian (x, y) and polar coordinates (r, θ) of a point P should be obvious.

$x = r\cos\theta$, and $y = r\sin\theta$, $r^2 = x^2 + y^2$ hence, $\cos\theta = x/r$, $\sin\theta = y/r$, $\tan\theta = y/x$ and $r = \sqrt{x^2 + y^2}$.

Note that it is conventional to take $r \geq 0$. When sketching a curve, given by a polar equation $r = \mathrm{f}(\theta)$ negative values of $\mathrm{f}(\theta)$ may occur for some values of θ. In such cases we use the corresponding positive value of $\mathrm{f}(\theta)$ and add π to (or subtract π from) the value of θ . This is perfectly valid since the same point is given by (r, θ) and $(-r, \theta + \pi)$. The addition of $2n\pi$ to θ, for any integer value of n, leaves the position of the point unaltered and so we choose a principal value of θ in the range $-\pi < \theta \leq \pi$.

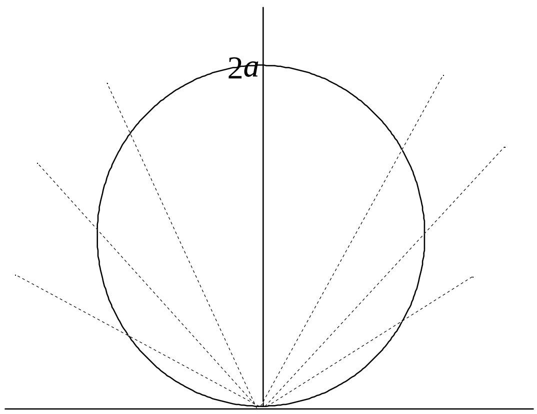
The polar coordinates of a point P may be compared with the modulus and argument of the complex number represented by the point P in the complex plane.

To sketch a curve from a given polar equation we either work out values of r corresponding to simple values of θ such as multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ or by converting to a cartesian equation.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	0	a	$a\sqrt{2}$	$a\sqrt{3}$	$2a$	$a\sqrt{3}$	$a\sqrt{2}$	a	0

Ex. Sketch the curve $r = 2a\sin\theta$

Method 1. We draw up a table of values.



Clearly for values of θ from π to 2π r will take the corresponding negative values so describing the curve twice.

Method 2. $r = 2a\sin\theta \Rightarrow \sqrt{x^2 + y^2} = \frac{2ay}{\sqrt{x^2+y^2}} \Rightarrow x^2 + y^2 - 2ay = 0$ which we may recognise to be the equation of a circle centre $(0, a)$ and radius a.

Sometimes we can do it by simple reasoning rather than working out values. In the case above we could argue as follows:

As θ increases from 0 to $\pi/2$, r increases from 0 to $2a$.

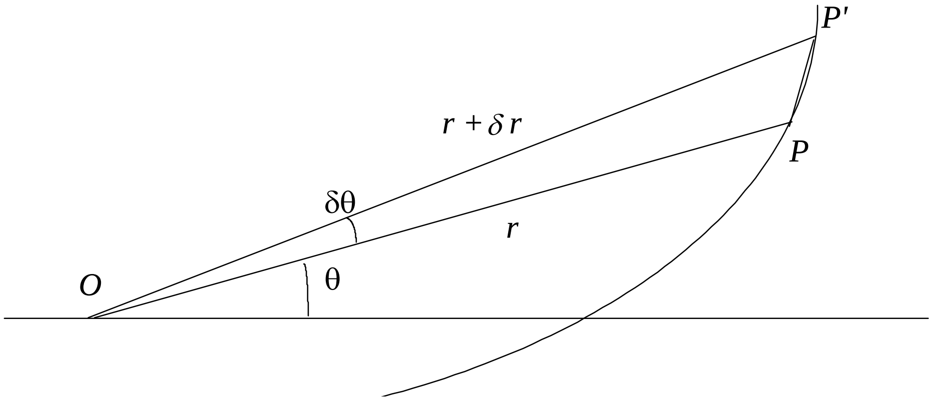
as θ increases from $\pi/2$ to π, r decreases from $2a$ to 0

This certainly tells us that we have a closed curve and one or two actual points should convince us that it is a circle.

Unless the curve retraces itself when $r < 0$ as in the case above, it is usual to draw such portion with a dotted line.

Areas in Polar Coordinates

Consider a curve with the polar equation $r = \mathrm{f}(\theta)$ and let P and P' be adjacent points on the curve with polar coordinates (r, θ) and $(r+\delta r, \theta+\delta\theta)$. The curve being such that both r and θ are increasing as we move from P to P'.



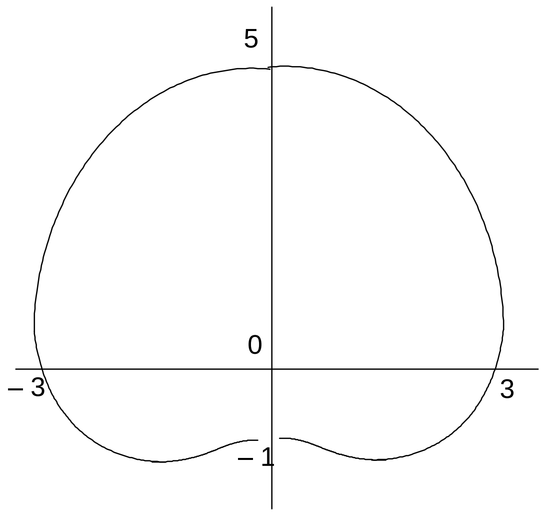
The area δA swept out by the radius vector in moving from OP to OP' is approximated by the area of the triangle OPP' i.e. $\delta A \approx \frac{1}{2}r(r+\delta r)\delta\theta$ since $\sin\delta\theta \approx \delta\theta$ for small angles . Ignoring the product of small quantities this gives $\delta A \approx \frac{1}{2}r^2\delta\theta$ and in the limit as $P \to P'$ and both δr and $\delta\theta \to 0$

$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{1}{2}r^2$ and so, area swept out as θ increases from α to β is $\int_\alpha^\beta \frac{1}{2}r^2\,\mathrm{d}\theta$

Ex. Find the area of the region enclosed by the curve $r = 3 + 2\sin\theta$

We first sketch the curve.

r has a maximum value of 5 when $\theta = \frac{1}{2}\pi$ and a minimum of 1 when $\theta = \frac{3}{2}\pi$ and $r = 3$ when $\theta = 0$ or π so we have a kidney shaped curve as shown.



$$\text{Area} = \int_0^{2\pi} \tfrac{1}{2}r^2\,\mathrm{d}\theta = 2 \times \tfrac{1}{2}\int_0^{\pi}(9 + 12\sin\theta + 4\sin^2\theta)\,\mathrm{d}\theta$$

making use of symmetry.

$$= \int_0^{\pi}(9 + 12\sin\theta + 2 - 2\cos 2\theta)\,\mathrm{d}\theta$$

$$= [11\theta - 12\cos\theta - \sin 2\theta]_0^{\pi} = 11\pi$$

The Inverse Trigonometric Functions

You will remember that for a function to have an inverse it must be one to one, hence, if the trigonometric functions are to have inverse functions defined, we must restrict their domains so that they become one to one functions. The usual restrictions are:

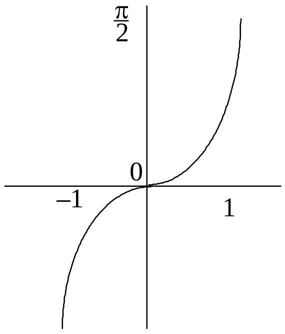
for $\sin x \quad -\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$ for $\cos x \quad 0 \le x \le \pi$ and for $\tan x \quad -\frac{1}{2}\pi < x < \frac{1}{2}\pi$

Note the strict inequality in the last case which is necessary since $\tan x$ is not defined for $x = \pm\frac{1}{2}\pi$

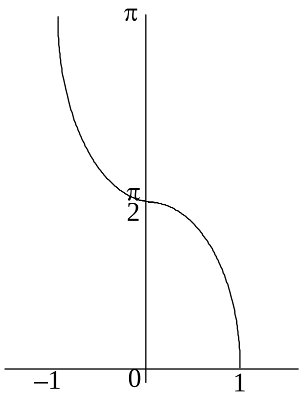
The inverse functions are denoted by the prefix 'arc', e.g. arcsin, arccos, arctan etc. or by using the index – 1, e.g. $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ etc. we thus have $\sin^{-1}x = y \Rightarrow \sin y = x$ for $-1 \le x \le 1$

$\cos^{-1}x = y \Rightarrow \cos y = x$ for $-1 \le x \le 1$ and $\tan^{-1}x = y \Rightarrow \tan y = x$ for all real x

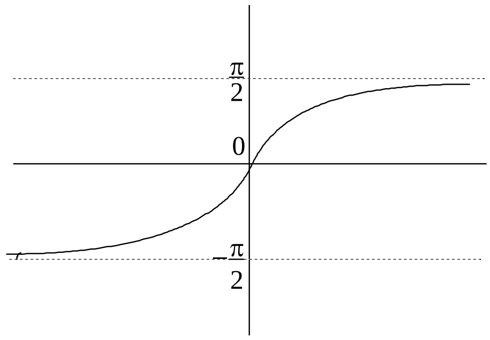
The graphs of the inverse trigonometric functions are easily obtained by reflecting the graphs of the trigonometric functions in the line $y = x$ providing equal scales are used on both axes.



$y = \sin^{-1}x$



$y = \cos^{-1}x$



$y = \tan^{-1}x$

Integration using inverse functions

We first consider the derivatives of the inverse trigonometric functions.

$y = \arcsin x \Rightarrow x = \sin y \Rightarrow 1 = \cos y\frac{dy}{dx}$ (implicit differentiation with respect to x)

so $\dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\pm\sqrt{1-\sin^2 y}} = \dfrac{1}{\pm\sqrt{1-x^2}}$

but from the graph of $\arcsin x$ we can see that it has a positive gradient throughout its domain hence we take only the positive root in the denominator to give $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

Alternative justification for taking only the positive root is that $y = \arcsin x \Rightarrow -\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and so $\cos x$ must be positive.

In a similar manner $y = \arccos x \Rightarrow \cos y = x \Rightarrow \frac{dy}{dx} = \frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}$

This time we see that $\arccos y$ has a negative gradient throughout

Finally $y = \arctan x \Rightarrow \tan y = x \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

Reversing these results we have

$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + c$, and $\int \frac{1}{1+x^2}dx = \arctan x + c$

More generally by making the substitution $x = a\sin x$ we can show that $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin(\frac{x}{a}) + c$

and putting $x = a\tan x$ we have $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan(\frac{x}{a}) + c$

Ex. Differentiate (i) $\sin^{-1}3x$ (ii) $\tan^{-1}\sqrt{x}$

(i) $\frac{d}{dx}(\arcsin 3x) = 3 \times \frac{1}{\sqrt{1-(3x)^2}}$ (function of a function) $= \dfrac{3}{\sqrt{1-9x^2}}$

(ii) $\frac{d}{dx}(\arctan\sqrt{x}) = \frac{1}{2\sqrt{x}} \times \frac{1}{1+x} = \frac{1}{2(1+x))\sqrt{x}}$

Ex. Evaluate (i) $\int_0^2 \frac{d\theta}{4+\theta^2}$ (ii) $\int_{1/3}^{2/3} \frac{dx}{\sqrt{4-9x^2}}$

(i) $\int_0^2 \frac{d\theta}{4+\theta^2} = \left[\frac{1}{2}\tan^{-1}\left(\frac{\theta}{2}\right)\right]_0^2 = \frac{1}{2}[\tan^{-1}1 - \tan^{-1}0] = \frac{1}{8}\pi$

(ii) $\int_{1/3}^{2/3} \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int_{1/3}^{2/3} \frac{dx}{\sqrt{\frac{4}{9}-x^2}} = \frac{1}{3}[\arcsin(\frac{3x}{2})]_{1/3}^{2/3} = \frac{1}{3}[\sin^{-1}1 - \sin^{-1}(\frac{1}{2})] = \frac{1}{3}(\frac{\pi}{2}-\frac{\pi}{6}) = \frac{\pi}{9}$

Harder Integrals

Integrals of functions of the form $\int \frac{P(x)dx}{ax^2+bx+c}$ where $P(x)$ is a polynomial in x

If $P(x)$ is of equal or higher degree than the denominator then we perform division until we have a remainder of lower degree so we need only consider that case. So suppose $P(x) = Ax + B$

If $A = 2a$ and $B = b$ then the numerator is the derivative of the denominator and we have a logartithmic integral. Otherwise we write $P(x) = k(2ax+b) - k'$ giving a logarithmic integral and an integral of the form $\int \frac{k'}{ax^2+bx+c}dx$. We therefore need only consider $\int \frac{1}{ax^2+bx+c}dx$

We do this by completing the square in the denominator to produce one of the following forms.

$\int \frac{dx}{X^2-A^2}$ $\int \frac{dx}{X^2+A^2}$ or $\int \frac{dx}{A^2-X^2}$ The first and third of these may now be done by using partial fractions to produce two logarithmic integrals whilst the second case gives an inverse tangent .

Integrals of the form $\int \frac{P(x)dx}{\sqrt{ax^2+bx+c}}$ Again, by similar techniques as before we can reduce the problem to two types of integral.

$\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx = 2\sqrt{ax^2+bx+c} + k$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ and again, by completing the square inside the square root we obtain one of the following forms:

$\int \frac{dx}{\sqrt{A^2-X^2}}$, $\int \frac{dx}{\sqrt{A^2+X^2}}$ or $\int \frac{dx}{\sqrt{X^2-A^2}}$

The first of these gives an inverse sine result. For the second by substituting $X = A\sec\theta$ we have $\frac{dx}{d\theta} = A\sec\theta\tan\theta$ so the integral becomes $\int \frac{A\sec\theta\tan\theta d\theta}{A\tan\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + k$

$$= \ln\left|\frac{X}{A} + \frac{\sqrt{X^2-A^2}}{A}\right| + k = \ln\left(X + \sqrt{X^2-A^2}\right) + k'$$

The third case is dealt with later in this book.

Maclaurin Series

Suppose a function $f(x)$ is expressible as a polynomial in x of degree n

i.e. $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, then the coefficients a_i ($i = 0$ to n) are all determined if the value of $f(x)$ and all of its derivatives are known for ssome particular value of x , for we then have a set of $n + 1$ equations in the $n+1$ unknowns $a_0, a_1, a_2, \dots, a_n$ which may be solved simultaneously. This is especially easy if the particular value of x is zero for then we immediately have

$$f(0) = a_0,\ f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + na_n x^{n-1} \Rightarrow f'(0) = a_1$$

$$f''(x) = 2a_2 + 2.3a_3 x + 3.4a_4 x^2 + \dots + (n-1)na_n x^{n-2} \Rightarrow f''(0) = 2a_2$$

and in general $f^{(r)}(0) = r!a_r \Rightarrow a_r = \frac{f^{(r)}(0)}{r!}$

thus $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots + \frac{f^{(n)}(0)}{n!}x^n$

To prove that this technique can be extended to an infinite series is beyond the scope of this course but we will accept it as true providing the infinite series converges to a finite limit. The infinite expansion of a function obtained in this way is called the MACLAURIN SERIES for the function.

Ex. Find the Maclaurin series for $(1+x)^n$ if n is negative or fractional.

Assume $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

Writing $f(x) = (1+x)^n$ we have $f(0) = 1$

$$f'(x) = n(1+x)^{n-1} \Rightarrow f'(0) = n$$

$$f''(x) = (n-1)n(1+x)^{n-2} \Rightarrow f''(0) = n(n-1)$$

and, in general $f^{(r)}(x) = n(n-1)(n-2)\dots(n-r+1)(1+x)^{n-r} \Rightarrow f^{(r)}(0) = n(n-1)(n-2)\dots(m-r+1)$

and so the Maclaurin expansion of $(1+x)^n$ is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots$$ providing this series is convergent.

Investigation of convergence is outside our syllabus but it is interesting to consider one standard test that will suffice for most of the series we shall meet.

First there is the fairly obvious requirement that the terms must tend to zero eventually,

i.e. $\lim_{r\to\infty} u_r = 0$, where u_r denotes the r -the term.

This however is not in itself a sufficient condition for convergence as can be seen by considering the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{r}$ which does not converge.

If however , we also have $\lim_{r\to\infty} |\frac{u_{r+1}}{u_r}| < 1$ then convergence is guaranteed.

Applying this to our series for $(1+x)^n$ we have $\frac{u_{r+1}}{u_r} = \frac{(n-r+1)x}{r} \to -x$ as $r \to \infty$

The condition for convergence is thus satisfied if $|x| < 1$ and clearly we also have $u_r \to 0$

Note that if n is a positive integer then the series terminates at the $(n+1)^{\text{th}}$ term and is identical to the series we have met earlier. In this case of course, there is no restriction on the value of x.

Ex. Find the first four terms in the expansions of (a) $(1+x)^{-1}$ (b) $(1+x)^{-2}$ (c) $(1+x)^{1/2}$ (d) $(1-\frac{x}{2})^{1/2}$ (e) $(3+2x)^{-1}$state also the range values of x for which your series is valid.

(a) $(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + .. = 1 - x + x^2 - x^3 + \dots$ for $|x| < 1$

(b) $(1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \dots = 1 - 2x + 3x^2 - 4x^3 + \dots$ for $|x| < 1$

(c) $(1+x)^{1/2} = 1 + (\frac{1}{2})x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \dots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$ for $|x| < 1$

(d) $(1-\frac{x}{2})^{1/2} = 1 + (\frac{1}{2})(-\frac{x}{2}) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(-\frac{x}{2})^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(-\frac{x}{2})^3 + \dots = 1 - \frac{1}{4}x - \frac{1}{32}x^2 - \frac{1}{128}x^3 + .$ for $|x| < 2$

(e) $(3+2x)^{-1} = 3^{-1}(1+\frac{2x}{3})^{-1} = \frac{1}{3}\left[1 + (-1)(\frac{2x}{3}) + \frac{(-1)(-2)}{2!}(\frac{2x}{3})^2 + \frac{(-1)(-2)(-3)}{3!}(\frac{2x}{3})^3 + \dots\right]$

$= \frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \dots$ for $|x| < \frac{3}{2}$

Note especially the method of the last example.

In general, $(a+bx)^n = a^n(1+\frac{bx}{a})^n$ if $a \neq 1$ and is valid for $|x| < \frac{a}{b}$

The binomial series is also very useful for estimation.

Ex. Estimate the value of $\sqrt[3]{997}$ giving your answer correct to 6 decimal places

$$\sqrt[3]{997} = (1000-3)^{1/3} = 10(1-0.003)^{1/3}$$

$$= 10\left(1+\tfrac{1}{3}(-0.003)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(-0.003)^2+\ldots\right)$$

$$= 10(1-0.001+0.000001-\ldots$$

It is clear that the next term will not affect the 6th decimal place so $\sqrt[3]{997} \approx 10\times 0.9989990 = 9.989990$

The following expansions are especially important and should be memorised

$(1+x)^{-1} = 1-x+x^2-x^3+\ldots$ $\qquad$ $(1-x)^{-1} = 1+x+x^2+x^3+\ldots$

$(1+x)^{-2} = 1-2x+3x^2-4x^3+\ldots$

What can we do with $(1+x)^n$ if x does not lie between ± 1 ?

Consider $*(1+x)^{-3}$ when x is large. We may write $(1+x)^{-3} = x^{-3}(1+\frac{1}{x})^{-3}$ and since x is large, $\frac{1}{x}$ must be small and so we have $\frac{1}{x^3}\left[1-\frac{3}{x}+\frac{6}{x^2}-\frac{10}{x^3}+\ldots\right] = \frac{1}{x^3}-\frac{3}{x^4}+\frac{6}{x^5}-\frac{10}{x^6}+\ldots$ for $|x|>1$

i.e. we have an expansion in descending powers of x

Ex. Express $\frac{x}{(1-x)(2-x)}$ as a series of (i) ascending (ii) descending powers of x

(i) We use partial fractions $\frac{x}{(1-x)(2-x)} = \frac{1}{1-x}-\frac{2}{2-x} = (1-x)^{-1}-2(2-x)^{-1}$

now $(1-x)^{-1} = 1+x+x^2+x^3+\ldots$ and $(2-x)^{-1} = \frac{1}{2}(1-\frac{x}{2})^{-1} = \frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots\right)$

hence, $\frac{x}{(1-x)(2-x)} = \left(1+x+x^2+x^3+.\right)-\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+.\right) = \frac{x}{2}+\frac{3x^2}{4}+\frac{7x^3}{8}+\ldots$

(ii) For large x, $(1-x)^{-1} = -\frac{1}{x}(1-\frac{1}{x})^{-1} = -\frac{1}{x}\left(1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots\right) = -\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{x^4}-\ldots$

and $(2-x)^{-1} = -\frac{1}{x}(1-\frac{2}{x})^{-1} = -\frac{1}{x}\left(1+\frac{2}{x}+\frac{4}{x^2}+\frac{8}{x^3}\right) = -\frac{1}{x}-\frac{2}{x^2}-\frac{4}{x^3}-\frac{8}{x^4}-\ldots$

so $\frac{x}{(1-x)(2-x)} = \left(-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{x^4}-\ldots\right)-2\left(-\frac{1}{x}-\frac{2}{x^2}-\frac{4}{x^3}-\frac{8}{x^4}-\ldots\right) = \frac{1}{x}+\frac{3}{x^2}+\frac{7}{x^3}+\frac{15}{x^4}+\ldots$

Other important standard results. These should be committed to memory.

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots$ valid for all values of x.

$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+(-1)^r\frac{x^{2r+1}}{(2r+1)!}+\ldots$ valid for all values of x

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+(-1)^r\frac{x^{2r}}{(2r)!}+\ldots$ valid for all values of x

$\ln(1+x)) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{r+1}\frac{x^r}{r}+$ valid only for $-1<x\le 1$

Note particularly the last one as it is the only one without factorial denominators and the only one other than the binomial series that has a restricted range of values of x.

Complex Numbers again

De Moivre's Theorem

We have previously established that $(\cos\theta + j\sin\theta)(\cos\psi + j\sin\psi) = \cos(\theta+\psi) + j\sin(\theta+\psi)$
hence, if $\theta = \psi$ then $(\cos\theta + j\sin\theta)^2 = \cos 2\theta + j\sin 2\theta$
Similarly $(\cos\theta + j\sin\theta)^3 = (\cos\theta + j\sin\theta)^2(\cos\theta + j\sin\theta) = \cos 3\theta + j\sin 3\theta$
which suggests the general result $(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$
This result is known as DE MOIVRE'S THEOREM after Abraham de Moivre (1667-1754)
Formal proof is left as an exercise.

Ex. If $z = \cos\theta + j\sin\theta,$ find the value of $z^5 + z^{-5}$
We have $z^5 = (\cos\theta + j\sin\theta)^5 = \cos 5\theta + j\sin 5\theta$
and $z^{-5} = (z^{-1})^5 = (\cos(-\theta) + j\sin(-\theta))^5 = \cos(-5\theta) + j\sin(-5\theta) = \cos 5\theta - j\sin 5\theta$
Hence, $z^5 + z^{-5} = 2\cos 5\theta$

Ex. Evaluate $(\cos\frac{\pi}{12} + j\sin\frac{\pi}{12})^4$
By de Moivre's theorem $(\cos\frac{\pi}{12} + j\sin\frac{\pi}{12})^4 = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$

Ex Express $\cos 4\theta$ in terms of $\cos\theta$

$$\begin{aligned}
\cos 4\theta = \mathrm{Re}(\cos 4\theta + j\sin 4\theta) &= \mathrm{Re}(\cos\theta + j\sin\theta)^4 \\
&= \mathrm{Re}(\cos^4\theta + 4j\cos^3\theta\sin\theta + 6j^2\cos^2\theta\sin^2\theta + 4j^3\cos\theta\sin^3\theta + j^4\sin^4\theta) \\
&= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\
&= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\
&= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}$$

Can we form a complex power of a real number? In particular, what meaning can we attach to e^z if z is complex?
First let us consider a purely imaginary power $\mathrm{e}^{j\theta}$ We may assign a meaning to this by simply requiring that it has a series expansion, thus:

$$\begin{aligned}
\mathrm{e}^{j\theta} = 1 + \boldsymbol{j}\theta + \tfrac{1}{2!}(\boldsymbol{j}\theta)^2 + \tfrac{1}{3!}(j\theta)^3 + \ldots &= 1 + \boldsymbol{j}\theta - \tfrac{1}{2}\theta^2 - \tfrac{1}{3!}\boldsymbol{j}\theta^3 + \tfrac{1}{4!}\theta^4 + \ldots \\
&= \left(1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - \ldots\right) + \boldsymbol{j}\left(\theta - \tfrac{\theta^3}{3!} + \tfrac{\theta^5}{5!} - \ldots\right) = \cos\theta + \mathrm{j}\sin\theta
\end{aligned}$$

Hence, for the general complex power we have $\mathrm{e}^z = \mathrm{e}^{x+jy} = \mathrm{e}^x \times \mathrm{e}^{jy} = \mathrm{e}^x(\cos y + j\sin y)$ so e^x is the modulus and y is the argument of the complex number e^z

The n-th roots of unity

$$(\cos\tfrac{p}{q}\theta + \mathrm{j}\sin\tfrac{p}{q}\theta)^q = \cos p\theta + \mathrm{j}\sin p\theta = (\cos\theta + \mathrm{j}\sin\theta)^p$$

so $\cos\frac{p}{q}\theta + \mathrm{j}\sin\frac{p}{q}\theta$ is a q -th root of $(\cos\theta + \mathrm{j}\sin\theta)^p$ i.e. a value of $(\cos\theta + \mathrm{j}\sin\theta)^{\frac{p}{q}}$
thus, for rational n, $\cos n\theta + \mathrm{j}\sin n\theta$ is one value of $(\cos\theta + \mathrm{j}\sin\theta)^n$
The reason why we only say "one value of" will soon be apparent.
$z^n = 1 \Rightarrow (\cos\theta + \mathrm{j}\sin\theta)^n = 1$ where $z = \cos\theta + \mathrm{j}\sin\theta \Rightarrow \cos n\theta + \mathrm{j}\sin n\theta = 1 \Rightarrow n\theta = 2k\pi$ for $k = 0, 1, 2, \ldots$
Thus $z = \cos(\frac{2k\pi}{n}) + \mathrm{j}\sin(\frac{2k\pi}{n})$ for integral values of k from 0 to $n-1$ are the distinct n-th roots of unity.
Taking $k > n-1$ merely gives a repetition of an earlier root .

Ex. Solve $z^3 - 1 = 0$
By the result just obtained the solutions are the cube roots of unity
i.e. $z = \cos 0 + \mathrm{j}\sin 0,\ \cos\frac{2\pi}{3} + \mathrm{j}\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + \mathrm{j}\sin\frac{4\pi}{3}$ or $1,\ \frac{1}{2}(-1 \pm \mathrm{j}\sqrt{3})$

Ex Solve $z^4 + z^3 + z^2 + z + 1 = 0$
Since $(z-1)(z^4 + z^3 + z^2 + z + 1) = z^5 - 1$ it follows that the required solutions are the complex fifth roots of unity, i.e. $z = \cos\frac{2\pi}{5} + \mathrm{j}\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} + \mathrm{j}\sin\frac{4\pi}{5},\ \cos\frac{6\pi}{5} + \mathrm{j}\sin\frac{6\pi}{5},\ \cos\frac{8\pi}{5} + \mathrm{j}\sin\frac{8\pi}{5}$
Again note that if we write $\omega = \cos\frac{2\pi}{5} + \mathrm{j}\sin\frac{2\pi}{5},$ then the other roots are $\omega^2,\ \omega^3$ and ω^4
We may also note that since $\omega^r\omega^{n-r} = \omega^n = 1,$ the complex roots occur in conjugate pairs.

The equation $z^n = c$

Let c be any complex number. If u is any one root of $z^n = c$ then we may write $(\frac{z}{u})^n = 1$
Hence, all roots are given by $u(\cos\frac{2k\pi}{n} + \mathrm{j}\sin\frac{2k\pi}{n})$ for $k = 0, 1, 2, ..., n-1$
Now write c in the form $r(\cos\theta + \mathrm{j}\sin\theta)$ where $r = |c|$ If $r^{\frac{1}{n}}$ denotes the real n-th root of r then $r^{\frac{1}{n}}[\cos\frac{\theta}{n} + \mathrm{j}\sin\frac{\theta}{n}]$ is an n-th root of c
Taking u to be this root, the n-th roots of c are given by $r^{\frac{1}{n}}[\cos(\frac{\theta+2k\pi}{n}) + \mathrm{j}\sin(\frac{\theta+2k\pi}{n})]$ for $k = 0$ to $n-1$

Ex. Solve the equation $z^5 = 32\mathrm{j}$

$r = 32$ hence $r^{\frac{1}{5}} = 2\mathrm{j} = \cos\frac{\pi}{2} + \mathrm{j}\sin\frac{\pi}{2}$ hence $z = 2\left[\cos\frac{\frac{\pi}{2}+2k\pi}{5} + \mathrm{j}\sin\frac{\frac{\pi}{2}+2k\pi}{5}\right]$ for $k = 0$ to 4

i.e. $z = 2(\cos\frac{\pi}{10} + \mathrm{j}\sin\frac{\pi}{10}),\ 2(\cos\frac{\pi}{2} + \mathrm{j}\sin\frac{\pi}{2}),\ 2(\cos\frac{9\pi}{10} + \mathrm{j}\sin\frac{9\pi}{10}),\ 2(\cos\frac{13\pi}{10} + \mathrm{j}\sin\frac{13\pi}{10})$ and $2(\cos\frac{17\pi}{10} + \mathrm{j}\sin\frac{17\pi}{10})$

Application of de Moivre's theorem to series

Certain types of trigonometric series may be summed by making use of de Moivre's theorem.

Ex. Prove that $\sum_{r=1}^{n} \cos r\theta = \frac{\cos\frac{1}{2}(n+1)\theta \sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$ for $\theta \neq 2k\pi$, and find an expression for $\sum_{r=1}^{n} \sin r\theta$

Putting $z = \cos\theta + \mathrm{j}\sin\theta$ then $z^r = \cos r\theta + \mathrm{j}\sin r\theta$

so writing $C + \mathrm{j}S = \sum_{r=1}^{n} z^r$ we have $C = \sum_{r=1}^{n} \cos r\theta$ and $S = \sum_{r=1}^{n} \sin r\theta$

Now $C + \mathrm{j}S = \frac{z(1-z^n)}{1-z}$ by the sum of a geometric series

and $1 - z^n = 1 - \cos n\theta - \mathrm{j}\sin n\theta = 2\sin^2(\frac{n\theta}{2}) - 2\mathrm{j}\sin(\frac{n\theta}{2})\cos(\frac{n\theta}{2}) = -2\mathrm{j}\sin(\frac{n\theta}{2})[\cos(\frac{n\theta}{2}) + \mathrm{j}\sin(\frac{n\theta}{2})]$
so putting $n = 1$ we have $1 - z = -2\sin(\frac{\theta}{2})[\cos\frac{\theta}{2} + \mathrm{j}\sin\frac{\theta}{2}]$

Hence $C + \mathrm{j}S = \frac{(\cos\theta + \mathrm{j}\sin\theta)[2\mathrm{j}\sin\frac{n\theta}{2}(\cos\frac{n\theta}{2} + \mathrm{j}\sin\frac{n\theta}{2})]}{2\mathrm{j}\sin\frac{\theta}{2}(\cos\frac{\theta}{2} + \mathrm{j}\sin\frac{\theta}{2})}$ so equating real and imaginary parts

$C = \sum_{r=1}^{n} \cos r\theta = \frac{\cos\frac{1}{2}(n+1)\theta \sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$ and $S = \sum_{r=1}^{n} \sin r\theta = \frac{\sin\frac{1}{2}(n+1)\theta \sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$

Complex Numbers in Geometry

Simple Loci

By simple analogy with vector geometry we can see that $|z_1 - z_2|$ is the distance between the points representing z_1 and z_2 in the Argand diagram. Thus, if A, B and P represent the complex constants a and b and the complex variable z then

(i) $|z-a| = k$ where k is real, defines a circle centre A, radius $AP = k$

(ii) $|z-a| = |z-b|$ defines the perpendicular bisector of the line AB

(iii) $|z-a| = k|z-b|$ with $k \neq 1$ is a circle with centre dividing AB externally in the ratio $k^2 : 1$

(iv) $\arg(z-a) = \theta$ with θ real defines the half-line from A, excluding the point A itself , proceeding in the direction θ measured from the x-axis

(v) $\arg\left(\frac{z-a}{z-b}\right) = \theta$ defines a circular arc from A to B such that $< APB = \theta$ measured from BP to AP i.e. θ is the angle of AP to the x-axis minus the angle of BP to the x-axis.

We prove the least obvious one which is (iii)

Let $z = x + \mathrm{j}y$, $a = a_1 + \mathrm{j}a_2$ and $b = b_1 + \mathrm{j}b_2$ then $|x - a_1 + \mathrm{j}(y - a_2)| = k|x - b_1 + \mathrm{j}(y - b_2)|$

$$\Rightarrow (x-a_1)^2 + (y-a_2)^2 = k^2\left[(x-b_1)^2 (y-b_2)^2\right]$$

$$\Rightarrow x^2 + y^2 - 2a_1x - 2a_2y + a_1^2 + a_2^2 = k^2(x^2 + y^2 - 2b_1x - 2b_2y + b_1^2 + b_2^2)$$

$$\Rightarrow (k^2-1)(x^2+y^2) + 2(a_1 - k^2b_1)x + 2(a_2 - k^2b_2)y + k^2(b_1^2 + b_2^2) - (a_1^2 + a_2^2) = 0$$

dividing by (k^2-1)(since $k \neq 1$) gives an equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ which we know to be the equation of a circle with centre at the point $\frac{k^2b_1 - a_1}{k^2-1} + \mathrm{j}\frac{k^2b_2 - b_1}{k^2-1} = \frac{k^2b - a}{k^2-1}$ which is the point dividing AB externally in the ratio $k^2 : 1$

Matrices Again

Inverse of a 3 x 3 matrix

Consider $\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ and let $\triangle = \det \mathbf{M} = \mathbf{a}.(\mathbf{b} \times \mathbf{c}) \neq 0$.

Let $\mathbf{L}$ be a matrix such that $\mathbf{LM} = \mathbf{I}$ then $\mathbf{Mr} = \mathbf{i} \Rightarrow \mathbf{LMr} = \mathbf{Li} \Rightarrow \mathbf{r} = \mathbf{Li}$ (the first column of $\mathbf{L}$)
Similarly $\mathbf{Lj}$ and $\mathbf{Lk}$ are the second and third columns of $\mathbf{L}$ respectively.

$\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{i} \Rightarrow x = \frac{\mathbf{i}.(\mathbf{b}\times\mathbf{c})}{\mathbf{a}.(\mathbf{b}\times\mathbf{c})} = \frac{A_1}{\triangle},\ y = \frac{B_1}{\triangle},\ z = \frac{C_1}{\triangle}$ so the first column of $\mathbf{L}$ is $\frac{1}{\triangle}\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix}$

Replacing $\mathbf{i}$ by $\mathbf{j}$ and then $\mathbf{k}$ in the above then leads to $\mathbf{L} = \frac{1}{\triangle}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

The matrix $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$ is the ADJUGATE or ADJOINT of $\mathbf{M}$ and is formed by replacing each element of $\mathbf{M}$ by its cofactor and then transposing (i.e. Changing rows into columns and v.v.)
It is denoted by adj$\mathbf{M}$, hence, the inverse of $\mathbf{M}$, $\mathbf{M}^{-1} = \frac{1}{\triangle}\,\mathrm{adj}\mathbf{M}$

Ex Find the inverse of the matrix $\mathbf{M} = \begin{pmatrix} 4 & -5 & 3 \\ 3 & 3 & -4 \\ 5 & 4 & -6 \end{pmatrix}$

We first calculate the cofactors thus:

$A_1 = \begin{vmatrix} 3 & -4 \\ 4 & -6 \end{vmatrix} = -18 + 16 = -2,\ A_2 = -\begin{vmatrix} -5 & 3 \\ 4 & -6 \end{vmatrix} = 30 - 12 = -18,\ A_3 = \begin{vmatrix} -5 & 3 \\ 3 & -4 \end{vmatrix} = 20 - 9 = 11$

Similarly $B_1 = -(-18+20) = -2,\ B_2 = -24 - 15 = -39,\ B_3 = -(-16-9) = 25$
and $C_1 = 12 - 15 = -3,\ C_2 = -(16+25) = -41,\ C_3 = 12 + 15 = 27$
hence, $\triangle = 4 \times (-2) + 3 \times (-18) + 5 \times 11 = -7$

$\mathrm{adj}\mathbf{M} = \begin{pmatrix} -2 & -18 & 11 \\ -2 & -39 & 25 \\ -3 & -41 & 27 \end{pmatrix}$ so $\mathbf{M}^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 18 & -11 \\ 2 & 39 & -25 \\ 3 & 41 & -27 \end{pmatrix}$

Eigenvalues and eigenvectors

Definition: If $\mathbf{s}$ is a non-zero vector such that $\mathbf{Ms} = \lambda\mathbf{s}$, where $\mathbf{M}$ is a matrix and λ is a scalar, then $\mathbf{s}$ is called an EIGENVECTOR of $\mathbf{M}$ and the scalar λ is the corresponding EIGENVALUE
To find eigenvectors we need to solve the equation $\mathbf{Ms} = \lambda\mathbf{s}$
Now $\mathbf{Ms} = \lambda\mathbf{s} \Rightarrow \mathbf{Ms} - \lambda\mathbf{s} = \mathbf{0} \Rightarrow \mathbf{Ms} - \lambda\mathbf{Is} = \mathbf{0} \Rightarrow (\mathbf{M} - \lambda\mathbf{I})\,\mathbf{s} = \mathbf{0}$
For non-zero solutions we must have $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$. This is known as the CHARACTERISTIC EQUATION of $\mathbf{M}$. Det$(\mathbf{M} - \lambda\mathbf{I})$ expands to give a polynomial in λ, the CHARACTERISTIC POLYNOMIAL. (Note! Eigenvalues and eigenvectors are also known as characteristic values and vectors)

Ex Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{M} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

We form the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$ i.e. $(\frac{1}{2} - \lambda)(-\frac{1}{2} - \lambda) - \frac{3}{4} = 0$
$\Rightarrow -\frac{1}{4} + \lambda^2 - \frac{3}{4} = 0 \Rightarrow \lambda^2 = 1$ so $\lambda = \pm 1$ are the eigenvalues.

To find the corresponding eigenvectors we proceed as follows, solving $(\mathbf{M} - \lambda\mathbf{I})\,\mathbf{s} = \mathbf{0}$

If $\mathbf{s} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1$ then we require $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

i.e. $-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 0$ and $\frac{\sqrt{3}}{2}x - \frac{3}{2}y = 0 \Rightarrow x = \sqrt{3}\,y$ and $\sqrt{3}\,x = 3y$ which is the same as $x = \sqrt{3}\,y$

so the eigenvector is a vector along the line $x = \sqrt{3}\,y$ for example the vector $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

and $x = \sqrt{3}\,y$ is a POINTWISE INVARIANT LINE.

$\lambda = -1 \Rightarrow \begin{pmatrix} \frac{3}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \frac{3}{2}x + \frac{\sqrt{3}}{2}y = 0$ and $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 0$

i.e. $3x + \sqrt{3}\,y = 0$ and $\sqrt{3}\,x + y = 0$ so the other invariant line is $y = -\sqrt{3}\,x$ with eigenvector $\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

3 by 3 matrices are dealt with in a similar manner.

<u>Ex</u> Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{M} = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

The characteristic equation is $\begin{vmatrix} 3-\lambda & 0 & 2 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$

i.e. $(3-\lambda)(2-\lambda)(-1-\lambda) = 0$ so $\lambda = -1, 2$ or 3

$\lambda = -1 \Rightarrow \begin{pmatrix} 4 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{matrix} 4x + 2z = 0 \\ x + 3y + z = 0 \end{matrix}$ so $z = -2x$ and $x = 3y$, so eigenvector is $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix}$

$\lambda = 2 \Rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{matrix} x + 2z = 0 \\ x + z = 0 \\ -3z = 0 \end{matrix}$ so $x = z = 0$, $y =$ any so eigenvector is $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 3 \Rightarrow \begin{pmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{matrix} 2z = 0 \\ x - y + z = 0 \\ -4z = 0 \end{matrix}$ so $z = 0$, $x = y$ so eigenvector is $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

An important application of eigenvalues and eigenvectors is to express a square matrix $\mathbf{M}$ in the form $\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix making it very easy to obtain powers of a square matrix.

<u>Ex</u> Consider the matrix $\mathbf{M} = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ dealt with above. Express $\mathbf{M}$ in the form $\mathbf{PDP}^{-1}$ and hence find $\mathbf{M}^3$ and $\mathbf{M}^4$

Let $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ i.e. a diagonal matrix with the eigenvalues as diagonal elements.

Take $\mathbf{P} = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ -6 & 0 & 0 \end{pmatrix}$ i.e. columns are the eigenvectors. then $\det\mathbf{P} = 6$ so $\mathbf{P}^{-1} = \frac{1}{6}\begin{pmatrix} 0 & 0 & -1 \\ -6 & 6 & -2 \\ 6 & 0 & 3 \end{pmatrix}$

$$\mathbf{PDP}^{-1} = \tfrac{1}{6}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ -6 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ -6 & 6 & -2 \\ 6 & 0 & 3 \end{pmatrix} = \tfrac{1}{6}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ -6 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ -12 & 12 & -4 \\ 18 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{M}$$

i.e. $\mathbf{PDP}^{-1} = \mathbf{M}$

Thus $\mathbf{M}^3 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^3\mathbf{P}^{-1} = \tfrac{1}{6}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ -6 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ -6 & 6 & -2 \\ 6 & 0 & 3 \end{pmatrix}$

i.e. $\mathbf{M}^3 = \tfrac{1}{6}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ -6 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ -48 & 48 & -16 \\ 162 & 0 & 81 \end{pmatrix} = \begin{pmatrix} 27 & 0 & 14 \\ 19 & 8 & 11 \\ 0 & 0 & -1 \end{pmatrix}$

and $\mathbf{M}^4 = \tfrac{1}{6}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ -6 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ -6 & 6 & -2 \\ 6 & 0 & 3 \end{pmatrix} = \tfrac{1}{6}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ -6 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ -96 & 96 & -32 \\ 486 & 0 & 243 \end{pmatrix} = \begin{pmatrix} 81 & 0 & .40 \\ 65 & 16 & 35 \\ 0 & 0 & 1 \end{pmatrix}$

The Cayley-Hamilton Theorem

Continuing with the above matrix, the characteristic equation was $(\lambda+1)(\lambda-2)(\lambda-3)=0$

i.e. $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$

$\mathbf{M}^2 = \begin{pmatrix} 9 & 0 & 4 \\ 5 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ and so $\mathbf{M}^3 - 4\,\mathbf{M}^2 + \mathbf{M} + 6\,\mathbf{I} = \begin{pmatrix} 27 & 0 & 14 \\ 19 & 8 & 11 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 36 & 0 & 16 \\ 20 & 16 & 12 \\ 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

i.e. $\mathbf{M}^3 - 4\mathbf{M}^2 + \mathbf{M} + 6\mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$

This illustrates the Cayley-Hamilton theorem which states that a square matrix always satisfies its own characteristic equation.

The main use of this theorem is to find higher powers of a square matrix.

Example With $\mathbf{M}$ as above, find $\mathbf{M}^6$

Solution

We have $\mathbf{M}^3 - 4\mathbf{M}^2 + \mathbf{M} + 6\,\mathbf{I} = \mathbf{0} \Rightarrow \mathbf{M}^3 = 4\mathbf{M}^2 - \mathbf{M} - 6\mathbf{I} \Rightarrow \mathbf{M}^4 = 4\mathbf{M}^3 - \mathbf{M}^2 - 6\mathbf{M}$

i.e. $\mathbf{M}^4 = 15\,\mathbf{M}^2 - 10\,\mathbf{M} - 24\,\mathbf{I}$

$\mathbf{M}^5 = 15\,\mathbf{M}^3 - 10\,\mathbf{M}^2 - 24\,\mathbf{M} = 50\,\mathbf{M}^2 - 39\,\mathbf{M} - 90\,\mathbf{I}$

so $\mathbf{M}^6 = 50\,\mathbf{M}^3 - 39\,\mathbf{M}^2 - 90\,\mathbf{M} = 161\,\mathbf{M}^2 - 140\,\mathbf{M} - 300\,\mathbf{I}$

i.e. $\mathbf{M}^6 = \begin{pmatrix} 1449 & 0 & 644 \\ 805 & 644 & 483 \\ 0 & 0 & 161 \end{pmatrix} - \begin{pmatrix} 420 & 0 & 280 \\ 140 & 280 & 140 \\ 0 & 0 & -140 \end{pmatrix} - \begin{pmatrix} 300 & 0 & 0 \\ 0 & 300 & 0 \\ 0 & 0 & 300 \end{pmatrix} = \begin{pmatrix} 729 & 0 & 364 \\ 665 & 64 & 343 \\ 0 & 0 & 1 \end{pmatrix}$

Hyperbolic Functions

We define the hyperbolic cosine, sine and tangent by

$\cosh x = \mathrm{ch}x = \frac{1}{2}(e^x + e^{-x})$; $\sinh x = \mathrm{sh}x = \frac{1}{2}(e^x - e^{-x})$ and $\tanh x = \mathrm{th}x = \frac{\mathrm{sh}x}{\mathrm{ch}x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

also $\mathrm{sech}x = \frac{1}{\cosh x}$; $\mathrm{cosech}x = \frac{1}{\sinh x}$ and $\coth x = \frac{1}{\tanh x}$

The following give some justification for the 'invention' of these functions.

(i) a number of integrals which otherwise cannot be obtained, are expressible in terms of them

(ii) A uniform chain hanging freely between two fixed points takes the form of a CATENARY with equation $y = k\cosh(\frac{x}{k})$ where k is constant, referred to suitable axes.

(iii) If $x = a\cosh\theta$ and $y = b\sinh\theta$ then

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \cosh^2\theta - \sinh^2\theta = 1 \Rightarrow \frac{1}{4}(e^\theta + e^{-\theta})^2 - \frac{1}{4}(e^\theta - e^{-\theta})^2 = 1$

so $x = a\cosh\theta$ and $y = b\sinh\theta$ are parametric equations for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

This last result indicates the close analogy with the circular functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ etc. In fact there is a hyperbolic identity corresponding to each trigonometric one.

We have already seen that $\cosh^2\theta - \sinh^2\theta = 1$.

Consider now $\sinh^2\theta + \cosh^2\theta = \frac{1}{4}(e^\theta - e^{-\theta})^2 + \frac{1}{4}(e^\theta + e^{-\theta})^2 = \frac{1}{2}(e^{2\theta} + e^{-2\theta}) = \cosh 2\theta$

Thus $\sin^2\theta + \cos^2\theta = 1$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$

but $\sinh^2\theta + \cosh^2\theta = \cosh 2\theta$ and $\cosh^2\theta - \sinh^2\theta = 1$

OSBORN'S RULE enables us to write down any hyperbolic identity from the corresponding trigonometric one: "Change cos to cosh, sin to sinh, and change the sign of any term involving the product of two sines"

Thus $\sin(A+B) = \sin A\cos B + \cos A\sin B \Rightarrow \sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$

but $\cos(A+B) = \cos A\cos B - \sin A\sin B \Rightarrow \cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$

and $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ translates to $\tanh 2A = \frac{2\tanh A}{1+\tanh^2 A}$

Note that $\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$ translates to $\frac{-\sinh^2 A}{\cosh^2 A} = -\tanh^2 A$

Ex. Write down and prove the hyperbolic identity corresponding to $1 + \tan^2 x = \sec^2 x$

By Osborn's rule the identity is $1 - \tanh^2 x = \mathrm{sech}^2 x$

Proof Left hand side $= 1 - (\frac{e^x - e^{-x}}{e^x + e^{-x}})^2 = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} = \frac{1}{\cosh^2 x} = \mathrm{sech}^2 x$

Equations involving hyperbolic functions may be solved either by using similar methods to those for trigonometric equations or by using the definitions of the hyperbolic functions as in the following example.

Ex Solve the equation $2\sinh x - \cosh x = 1$

$2\sinh x - \cosh x = 1 \Rightarrow (e^x -^{-x}) - \frac{1}{2}(e^x + e^{-x}) = 1 \Rightarrow e^x - 3e^{-x} = 2 \Rightarrow e^{2x} - 2e^x - 3 = 0$

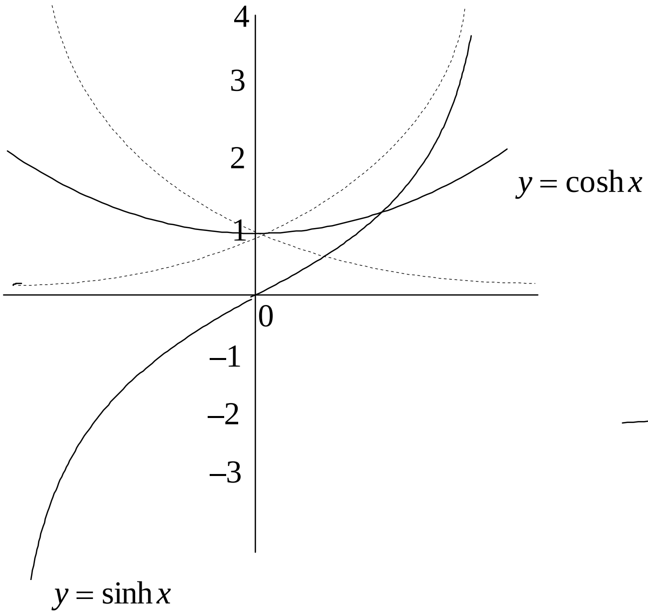
hence, $(e^x - 3)(e^x + 1) = 0$ but e^x cannot be negative so $e^x = 3$ is the only solution, i.e. $x = \ln 3$

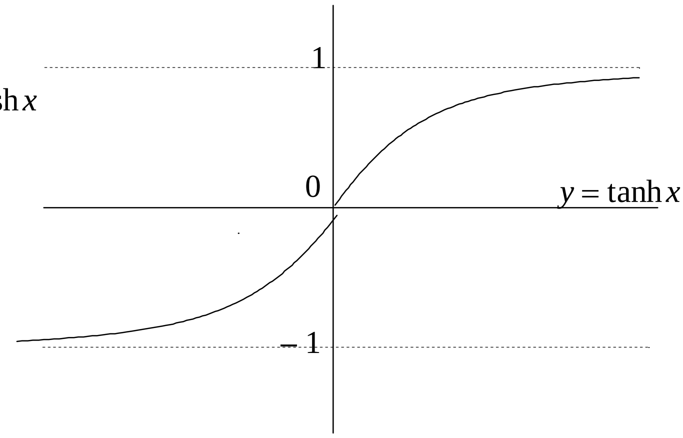
Differentiation and Integration

$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$, similarly $\frac{d}{dx}\cosh x = \sinh x$ and by the quotient rule $\frac{d}{dx}\tanh x = \mathrm{sech}^2 x$ with corresponding integral results.

The Inverse Functions

Consider first the graphs of $\sinh x$, $\cosh x$ and $\tanh x$ which are readily obtained from the graphs of e^x and e^{-x}





We can see from these graphs that $\sinh x$ is one to one with range and domain both equal to $\mathbb{R}$

We therefore define the inverse hyperbolic sine, $\sinh^{-1}x$, for all values of x to be that value of y such that $x = \sinh y$. Thus $y = \sinh^{-1}x \Rightarrow x = \sinh y = \frac{1}{2}(e^y - e^{-y}) \Rightarrow 2x = e^y - e^{-y}$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = x \pm \sqrt{x^2+1}$$

Since we must have $e^y > 0$ it follows that only the positive root is acceptable and so

$$\sinh^{-1}x = y = \ln\left(x + \sqrt{x^2+1}\right) \text{ for } x \in \mathbb{R}$$

$\cosh x$ is not one to one unles we restrict the domain, the usual restriction being to allow only $x > 0$

We can then define $\cosh^{-1}x$ to be the positive value of y such that $x = \cosh y$.

Thus $y = \cosh^{-1}x \Rightarrow x = \cosh y = \frac{1}{2}(e^y + e^{-y}) \Rightarrow e^{2y} - 2xe^y + 1 = 0 \Rightarrow e^y = x \pm \sqrt{x^2-1}$

Now $\left(x - \sqrt{x^2-1}\right)\left(x + \sqrt{x^2-1}\right) = x^2 - (x^2-1) = 1 \Rightarrow x - \sqrt{x^2-1} = \left[x + \sqrt{x^2-1}\right]^{-1}$

The domain of $\cosh^{-1}x$ is $\{x \in \mathbb{R} : x \geq 1\}$ and for $x > 1$ $x + \sqrt{x^2-1} > 1 \Rightarrow x - \sqrt{x^2-1} < 1$ and $e^y < 1 \Rightarrow y < 0$. But from our definition of $\cosh^{-1}x$ we must have $y \geq 0$ and so we cannot have $e^y = x - \sqrt{x^2-1}$. Hence $\cosh^{-1}x = y = \ln\{x + \sqrt{x^2-1}\}$

$\tanh x$ is clearly one to one for all real x and so we define $\tanh^{-1}x$ to be that value of y such that $x = \tanh y$ for $-1 < x < 1$. The logarithmic form for $\tanh^{-1}x$ is left as an exercise.

Derivatives and Integrals

$y = \sinh^{-1}x \Rightarrow x = \sinh y \Rightarrow \frac{dx}{dy} = \cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{x^2+1}}$

Only the positive root need be considered since $\sinh^{-1}x$ has a positive gradient for all values of x

Similarly $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$ (why only the positive root this time?) and $\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$

Verify these results for yourself. You could be asked to derive them in an examination.

Hence, $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}x + c = \ln A\left(x + \sqrt{x^2+1}\right)$ $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x + c = \ln A\left(x + \sqrt{x^2-1}\right)$

and $\int \frac{dx}{x^2-1} = \tanh^{-1}x + c = \frac{1}{2}\ln A\left(\frac{1+x}{1-x}\right)$ this one could also be done by partial fractions

What restrictions must be imposed on the values of x in these three integrals?

We can now complete our review of integrals of the form $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

By completing the square of the quadratic in the denominator we obtain one of the following:

$\int \frac{dx}{X^2+A^2} = \frac{1}{A}\arctan(\frac{X}{A})$; $\int \frac{dx}{X^2-A^2} = \frac{1}{2A}\ln k\left|\frac{X+A}{X-A}\right|$; $\int \frac{dx}{A^2-X^2} = \frac{1}{2A}\ln\left|\frac{A-X}{A+X}\right|$

where X has the form $x \pm b$ unless $ax^2 + bx + c < 0$ for all x in which case the integral does not exist.

Note that a logarithmic form is usually preferred to an inverse hyperbolic tangent.

$\int \frac{dx}{\sqrt{)A^2-X^2}} = \sin^{-1}(\frac{X}{A})$; $\int \frac{dx}{\sqrt{A^2+X^2}} = \sinh^{-1}(\frac{X}{A})$; $\int \frac{dx}{\sqrt{X^2-A^2}} = \cosh^{-1}(\frac{X}{A})$

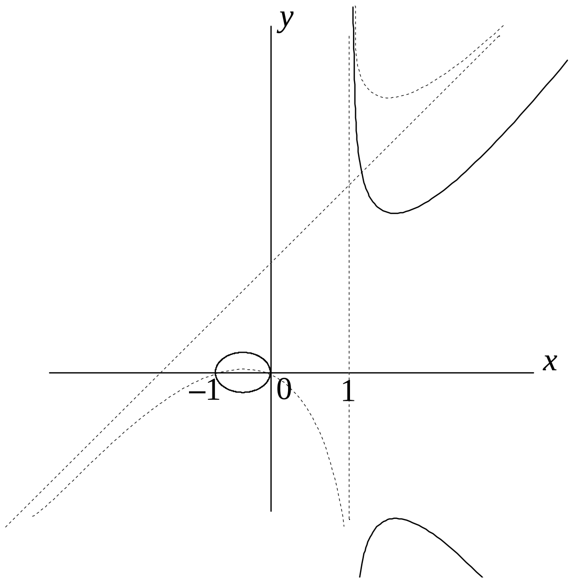
Curve Sketching Again

Most of the techniques were dealt with earlier. The remaining ones are (i) curves with equations of the form $y^2 = \mathrm{f}(x)$ and (ii) equations of the form $y = |\mathrm{f}(x)|$

(i) To sketch $y^2 = \mathrm{f}(x)$ we start by sketching $y = \mathrm{f}(x)$ and then plot the two square roots of each ordinate.

Ex. Sketch $y^2 = \frac{x(x+1)}{x-1}$

We first sketch $y = \frac{x(x+1)}{x-1}$ shown by a dotted line in the diagram below



We note that $x = 1$ is an asymptote.

There is no horizontal asymptote but writing the equation as $y = x + 2 + \frac{2}{x-1}$ we see that $y = x + 2$ is an asymptote. Considering how the curve approaches these asymptotes it is quite easy to sketch the curve.

Clearly, the curve can only exist for

$$-1 \le x \le 0 \text{ and } x > 1$$

Between $x = -1$ and $x = 0$ we must have a closed loop. Note that, in general, if $0 < y < 1$ then $\sqrt{y} > y$ while if $y > 1$ then $\sqrt{y} < y$ and the square root curve must intersect the original curve at any point where $y = 0$ or 1

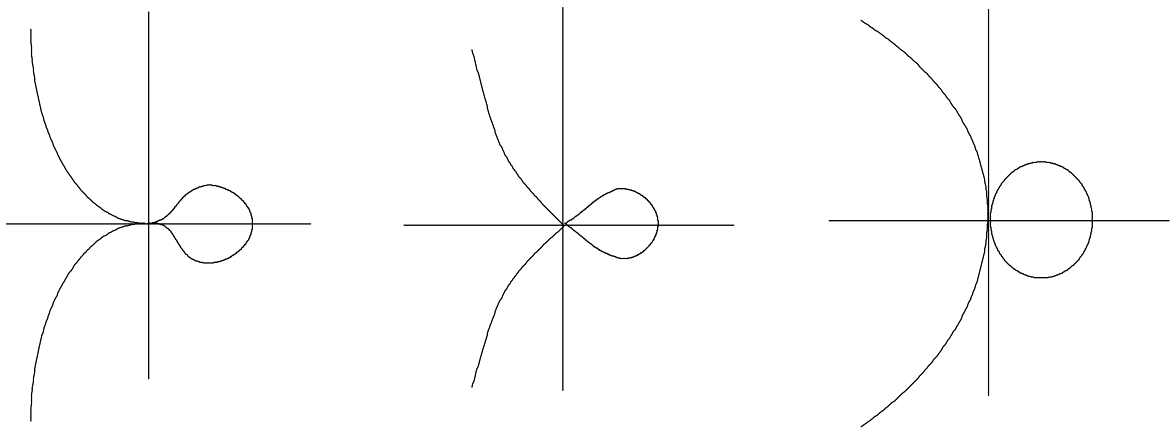
Ex. Sketch $y^2 = x^2(1 - x)$

This example is included to illustrate one final situation.. It is easy to sketch the graph of $y = x^2(1 - x)$ as it is simply a cubic, touching the x – axis at the origin and passing through $(1, 0)$

The square root curve obviously only exists for $x \le 1$ and we again must have a loop for $0 \le x \le 1$

The curves $y = x^2(1 - x)$ and $y^2 = x^2(1 - x)$ also intersect where $y = 1$, but how does the square root curve pass through the origin and how does it go off to infinity.

At the origin there are really just three possibilities



Since this is an important feature of the graph we consider how the curve behaves for small values of x

We have

$y^2 = x^2 - x^3 \Rightarrow y^2 \approx x^2$ for small x since the x^3 term may be neglected as being very small compared with $x^2 \Rightarrow y \approx \pm x$ for very small values of x so the middle diagram must be the correct one.

To deduce how the curve goes off to infinity consider $y = x^n$

(i) If $n = 1$ we have a straight line (ii) if $n > 1$ the line will get steeper as x increases

(iii) if $0 < n < 1$ the curve will get flatter as x increases

In the above example, since the dominant term for large x is x^3 it follows that the curve will get steeper as $x \to \pm\infty$ but not too rapidly. Can you sketch the final curve?

Graph of $y = |\mathrm{f}(x)|$

This is simply a matter of sketching $y = \mathrm{f}(x)$ and reflecting any part of the graph that is below the x-axis in that axis.

Inequalities

We encountered the solution of inequalities earlier . An alternative method is to use a sketch graph as follows.

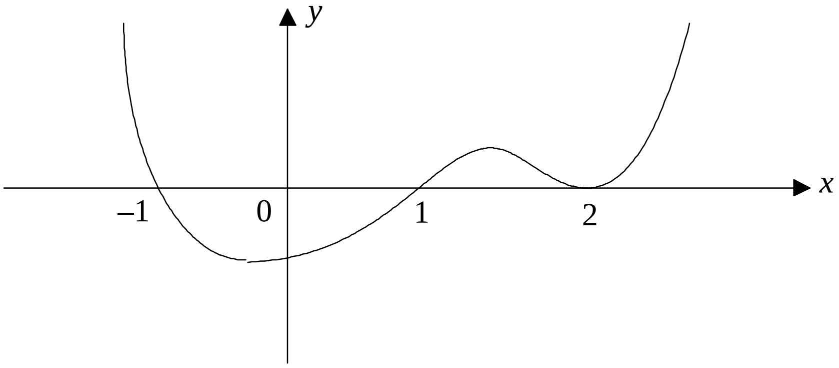
Ex. Solve the inequality $\frac{2x-1}{x-1} \le \frac{9}{x+1}$

Solution: This is equivalent to $(2x-1)(x+1)^2(x-1) \le 9(x-1)^2(x+1)$ Note that since we do not know whether the denominators are positive or negative we must multiply by the squares of the denominators to ensure that the inequality sign remains unchanged.

Thus we have $(2x-1)(x+1)^2(x-1) - 9(x-1)^2(x+1) \le 0 \Rightarrow (x+1)(x-1)[(2x-1)(x+1) - 9(x-1)]$ i.e. $(x+1)(x-1)(2x^2 - 8x + 8) \le 0 \Rightarrow 2(x+1)(x-1)(x-2)^2 \le 0$ so we plot $y = 2(x+1)(x-1)(x-2)^2$

This is easily sketched since it crosses the x- axis where $x = -1$ and 1 and touches the axis at $x = 2$

Clearly also, y is positive for large positive or negative values of x since the dominant term is x^4



Since we require $y \le 0$ we can easily see that the solution is $-1 \le x \le 1$ or $x = 2$

You should also be aware of the standard cartesian and parametric equations of the conics which are as follows:

Parabola: $y^2 = 4ax,\ x = at^2,\ y = 2at$ for a parabola symmetrical about x axis passing through origin.

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ x = a\cos\theta,\ y = b\sin\theta$ for symmetrically placed ellipse .

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ x = a\sec\theta,\ y = b\tan\theta$ or $x = a\cosh\theta,\ y = b\sinh\theta$

Rectangular Hyperbola: $xy = c^2,\ x = ct,\ y = \frac{c}{t}$

Further Vector Geometry - The Vector Product

The vector product $\mathbf{a}\times\mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ at an angle θ to each other, is defined to be the vector of magnitude $ab\sin\theta$ in a direction perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ form a right-handed set of vectors. i.e. An ordinary (right- handed) screw rotated from the direction of $\mathbf{a}$ to the direction of $\mathbf{b}$ would move in the direction of $\mathbf{a}\times\mathbf{b}$.

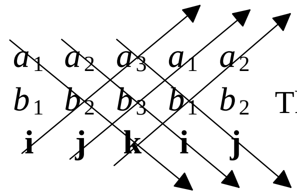
Thus $\mathbf{a}\times\mathbf{b}= (ab\sin\theta)\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector in the direction of $\mathbf{a}\times\mathbf{b}$

An immediate consequence of the definition is that $\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$ so vector multiplication is anti-commutative.

$\mathbf{a}\times\mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ is parallel to $\mathbf{b}$. In particular $\mathbf{a}\times\mathbf{a} = 0$ hence, $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$ whilst $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$, $\mathbf{j}\times\mathbf{i} = -\,\mathbf{k}$, $\mathbf{k}\times\mathbf{j} = -\,\mathbf{i}$ and $\mathbf{k}\times\mathbf{i} = -\,\mathbf{j}$

The distributive law $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$ can be shown to hold and so, in component form, with usual notation,

$\mathbf{a}\times\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = (a_2b_3 - b_2a_3)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

The following diagram is a convenient aid to memory 

The three triple products sloping downwards from left to right are positive and the three sloping upwards from left to right are negative.

Ex. Given $\mathbf{p} = \mathbf{i} + \mathbf{j}$ and $\mathbf{q} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find the components of the vector $\mathbf{p}\times\mathbf{q}$ in the directions of the axes. Find also the magnitude of the resolved part of $\mathbf{p}\times\mathbf{q}$ in the direction of $\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{p}\times\mathbf{q} = (2-0)\mathbf{i} + (0-2)\mathbf{j} + (-1-2)\mathbf{k} = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and the components are thus $2\mathbf{i}$, $-2\mathbf{j}$ and $-3\mathbf{k}$. The magnitude of the resolved part in the direction of $\hat{\mathbf{r}}$ is $(\mathbf{p}\times\mathbf{q}).\hat{\mathbf{r}} = \left|(2\mathbf{i}-2\mathbf{j}-3\mathbf{k}).\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}-\mathbf{k})\right|$

$$= \frac{1}{\sqrt{3}}\times 3 = \sqrt{3}$$

Ex. Find a unit vector perpendicular to both $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{a}\times\mathbf{b} = (-2+1)\mathbf{i} + (-4+2)\mathbf{j} + (2-8)\mathbf{k} = -\,\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$

Thus $-\,\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and so a unit vector is $\frac{1}{\sqrt{41}}(\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$

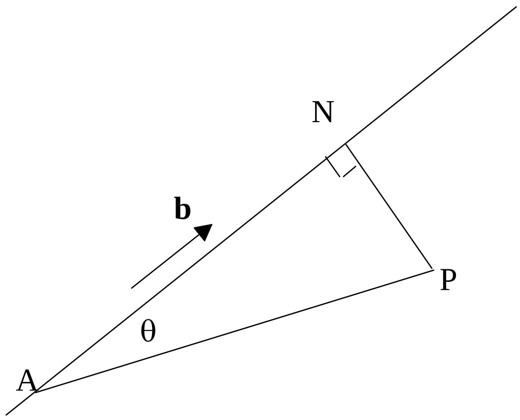
Applications of the vector product.

1. Line of intersection of two planes. The vector product of the normals to the planes gives us the direction vector for the line of intersection. Finding the coordinates of any one common point of the planes then enables us to write down the equation of the line of intersection.

Distance of a point from a line

Problem

To find the distance of the point $P(x, y, z)$ from the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$



Let A be any point on the line.

Required distance is PN

The vector product gives $\mathbf{AP}\times\mathbf{AN} = |\mathbf{AP}||\mathbf{AN}|\sin\theta$

so $\mathbf{AP}\times\hat{\mathbf{b}} = AP\sin\theta$ i.e. $PN = \left|\frac{\mathbf{AP}\times\mathbf{b}}{|\mathbf{b}|}\right|$

Special case. (2 dimensions)

Consider a point and a line in the plane $z = 0$

If equation of line is $ax + by + c = 0$ and P is the point $(x_1, y_1, 0)$ then we may take A to be $\left(0, -\frac{c}{b}, 0\right)$

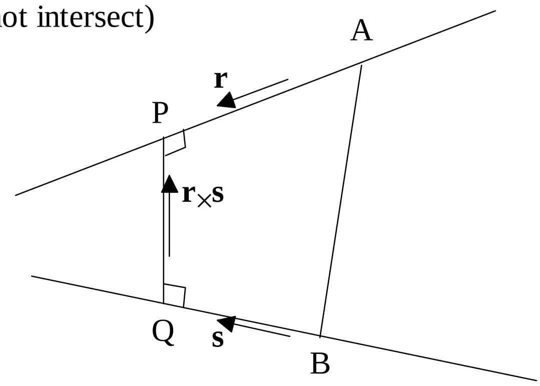
$$\mathbf{b} = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} \text{ and so } \mathbf{AP} \times \hat{\mathbf{b}} = \begin{pmatrix} x_1 \\ y_1 + \frac{c}{b} \\ 0 \end{pmatrix} \times \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ax_1 + by_1 + c \end{pmatrix}$$

hence, $PN = \left| \frac{\mathbf{AP} \times \mathbf{b}}{|\mathbf{b}|} \right| = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$

Triple Products.

There are two types of triple product.

Shortest distance between two skew lines (i.e. lines which do not intersect)



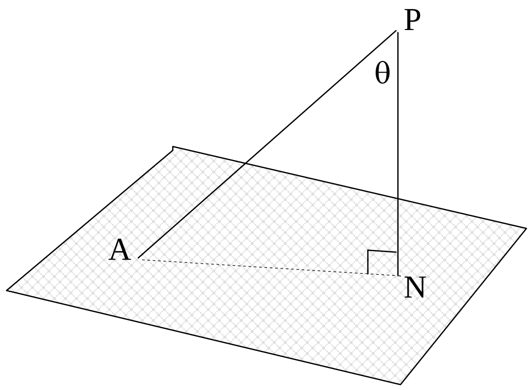
Let A and B be any points on each of the two lines

Shortest distance between the lines will be the length of the common perpendicular PQ hence, since APQ and BQP are right angles, PQ is the projection of AB on the common perpendicular to the two lines.

The direction of PQ is given by $\mathbf{r} \times \mathbf{s}$ where $\mathbf{r}$ and $\mathbf{s}$ are the direction vectors of the two lines, hence,

$PQ = \left| \frac{\mathbf{AB}.(\mathbf{r} \times \mathbf{s})}{|\mathbf{r} \times \mathbf{s}|} \right|$

Distance of a point from a plane



Problem

To find the distance of the point P from the plane

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

Solution

NP has direction $\mathbf{a} \times \mathbf{b}$ the normal to the plane.

$PN = |AP \cos\theta| = \left| \frac{\mathbf{AP}.\mathbf{PN}}{|\mathbf{PN}|} \right| = \left| \frac{(\mathbf{p} - \mathbf{a}).(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|} \right|$

Note. $\dfrac{(\mathbf{p} - \mathbf{a}).(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$ is positive if P is on the same side of the plane as the origin and negative if on opposite side.

If the equation of the plane is $ax + by + cz + d = 0$ and P is the point (x_1, y_1, z_1) a normal to the plane is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ so taking A to be the point } \begin{pmatrix} 0 \\ 0 \\ -\frac{d}{c} \end{pmatrix} \text{ we have } \mathbf{AP}.\mathbf{PN} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 + \frac{d}{c} \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax_1 + by_1 + cz_1 + d$$

hence, $PN = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$ note similarity to result for distance of point from line in 2 dimensions.

Scalar Triple Product

The second and third results above both involve an expression of the form $\mathbf{a}.\mathbf{b} \times \mathbf{c}$

We saw there that it measures the volume of a parallelepiped with edges defined by **a,b** and **c**

and is calculated as for a 3 by 3 determinant. i.e. $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note that a cyclic change of the letters does not affect the value but a non-cyclic change multiplies the product by –1.

Geometrical applications

An immediate application is to test whether two lines intersect. If they do then the shortest distance between them is zero and hence the scalar triple product in the previous work is zero. Conversely, if the scalar triple product is not zero then the lines do not intersect.

To test whether points A,B,C and D are coplanar.

If they are, then the parallelepiped defined by edges AB,AC and AD will have zero volume and so the scalar triple product $(\mathbf{b}-\mathbf{a}).[(\mathbf{c}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})]=0$

Testing for right or left-handedness

$\mathbf{a}.(\mathbf{b}\times\mathbf{c})>0 \Rightarrow \mathbf{a},\mathbf{b}$ and $\mathbf{c}$ form a right handed set, $\mathbf{a}.(\mathbf{b}\times\mathbf{c})<0 \Rightarrow \mathbf{a},\mathbf{b}$ and $\mathbf{c}$ form a left handed set

Remember that $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\mathbf{b}.(\mathbf{c}\times\mathbf{a})=\mathbf{c}.(\mathbf{a}\times\mathbf{b})$ but for example $\mathbf{a}.(\mathbf{c}\times\mathbf{b})=-\mathbf{a}.(\mathbf{b}\times\mathbf{c})$

Multivariable calculus

There are many situations where we are dealing with a function of more than one variable.
e.g. the vo**lu**me of a cylinder is given by $V = \pi r^2 h$ whilst a surface in 3-dimensional space will have an equation of the form $z = \mathrm{f}(x, y)$

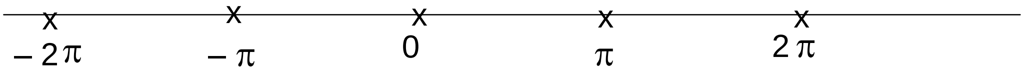
If $z = 2\cos x - y^2$ defines such a surface, letting x take a fixed value θ say, we have $z = 2\cos\theta - y^2$ which represents a curve in the y, z plane, i.e. a section of the surface perpendicular to the $x-$ axis whilst giving y a fixed value will produce a section of the surface perpendicular to the $y-$ axis.

If z has a fixed value we have a section parallel to the $x - y$ plane which is usually called a CONTOUR.

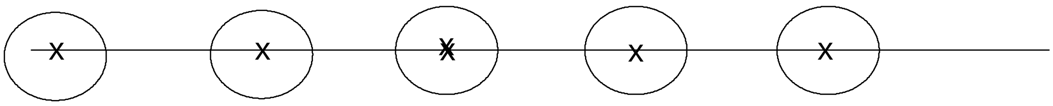
In this example the sections with x constant will be inverted parabolas, those with y constant will be cosine curves whilst the contours $z =$ constant will be of differenmt forms accoprding to the value of z.

The diagram shows a few contours for this surface.

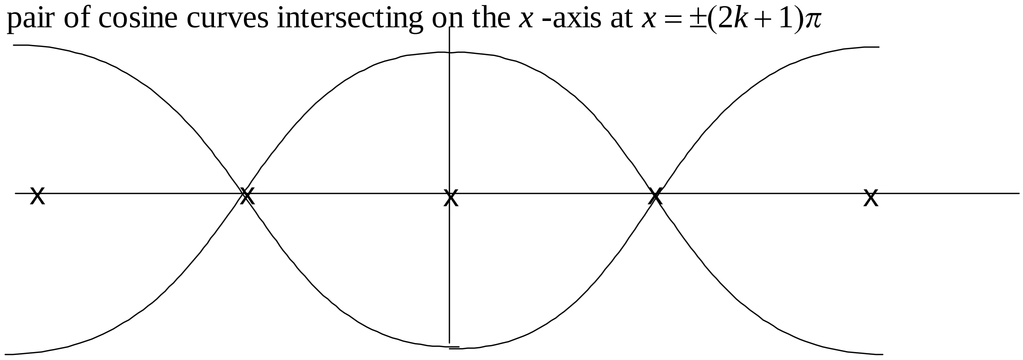
$z = 2$ gives a series of points at $x = \pm k\pi$



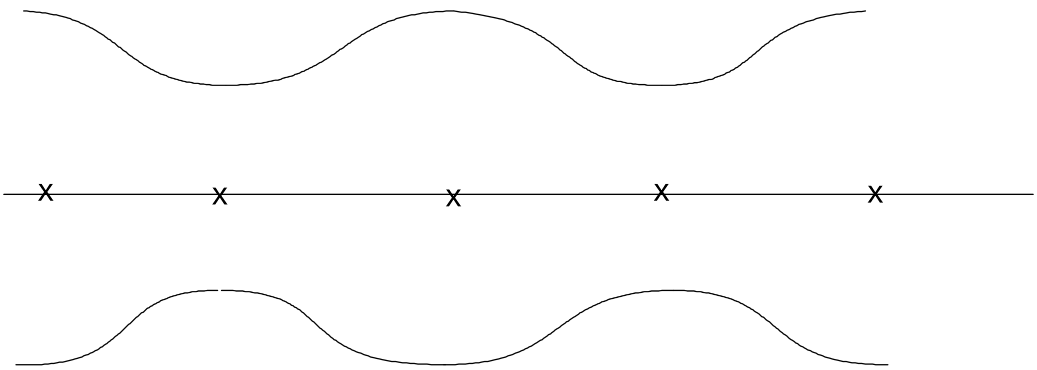
$0 < z < 2$ gives a series of circles centered at $x = \pm k\pi$



$k = -2$ gives a pair of cosine curves intersecting on the x -axis at $x = \pm(2k+1)\pi$



$k < -2$ and the curves do not intersect the axis.



Partial Differentiation

Suppose $z = \mathrm{f}(x, y)$ and let x increase by a small amount δx while y remains constant, then the increase in z is $\mathrm{f}(x + \delta x, y) - \mathrm{f}(x, y)$ and the average rate of change of z with respect to x is $\frac{\mathrm{f}(x+\delta x, y) - \mathrm{f}(x,y)}{\delta x}$

The limit of $\frac{\mathrm{f}(x+\delta x, y) - \mathrm{f}(x,y)}{\delta x}$ as $\delta x \to 0$, if it exists, is called the PARTIAL DERIVATIVE of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$ (Note! Be careful not to confuse the symbols ∂ and d)

Similarly $\frac{\partial z}{\partial y}$ is the partial derivative of z with respect to y, x remaining constant.

Ex Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = y^2 \ln x$

Treating y as a constant we have $\frac{\partial z}{\partial x} = y^2 \times \frac{1}{x} = \frac{y^2}{x}$
treating x as a constant $\frac{\partial z}{\partial y} = 2y \ln x$
Geometrically speaking, $\frac{\partial z}{\partial x}$ tells us the gradient of a section of the surface perpendicular to the $y-$ axis whilst $\frac{\partial z}{\partial y}$ gives the gradient of a section perpendicular to the x- axis .

Tangent Planes

Consider a surface $z = \mathrm{f}(x, y)$ and suppose that at the point $x = a,\ y = b$ on this surface the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are p and q respectively then the vectors $\begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix}$ must lie in the plane tangential to the surface at this point, hence, a normal to the plane is given by $\begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix} = \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix}$ and so the equation of the tangent plane is $-px - qy + z = -ap - bq + \mathrm{f}(a, b)$

Ex Find the equation of the tangent plane to the surface $z = \frac{x^3}{y^2}$ at the point given by $x = 4,\ y = -8$
$x = 4,\ y = -8 \Rightarrow z = 1.\ \frac{\partial z}{\partial x} = \frac{3x^2}{y^2} = \frac{48}{64} = \frac{3}{4}$ at $(4, -8, 1)$, $\frac{\partial z}{\partial y} = -\frac{2x^3}{y^3} = -\frac{128}{-512} = \frac{1}{4}$

So two vectors in the tangent plane are $\begin{pmatrix} 1 \\ 0 \\ \frac{3}{4} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix}$

A normal vector is $\begin{pmatrix} 1 \\ 0 \\ \frac{3}{4} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix}$ so tangent plane is $-\frac{3}{4}x - \frac{1}{4}y + z = -3 + 2 + 1 = 0$

i.e. tangent plane is $3x + y - 4z = 0$

Directional derivatives

To find the gradient on the surface $z = \mathrm{f}(x, y)$ at the point $\mathrm{A}(a, b, c)$ in the direction defined by the horizontal unit vector $\mathbf{\hat{u}} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ we put $\mathrm{d}x = \cos\alpha$ and $\mathrm{d}y = \sin\alpha$ in the equation of the tangent plane at A to give $\mathrm{d}z = \frac{\partial \mathrm{f}}{\partial x}\cos\alpha + \frac{\partial \mathrm{f}}{\partial y}\sin\alpha$, the partial derivatives being evaluated at A. Since this value of dz is the vertical change needed to get back to the tangent plane after a unit horizontal step, it is also a measure of the required gradient. This is known as the DIRECTIONAL DERIVATIVE in the direction of $\mathbf{\hat{u}}$.

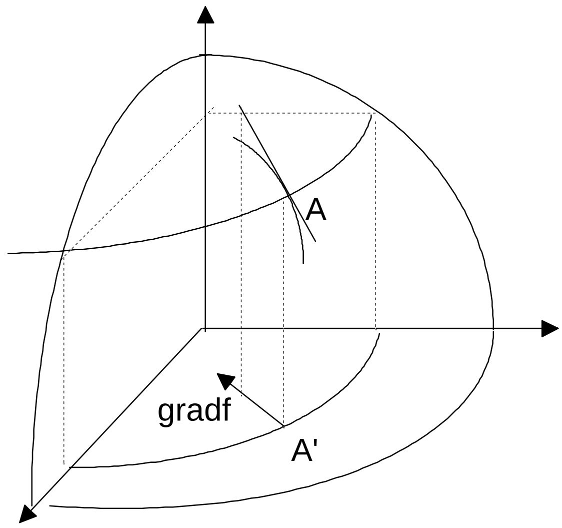
The vector grad f

The directional derivative can also be written as the scalar product $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\begin{pmatrix} \partial \mathrm{f}/\partial x \\ \partial \mathrm{f}/\partial y \end{pmatrix}$

The second of these vectors is called grad f, or ∇ f . ∇ is pronounced "nabla"

grad f is a vector normal to the contour through A and is also the maximum gradient on the surface.
Summarising, if $\mathrm{A}(a, b, c)$ is a point on the surface $z = \mathrm{f}(x, y)$ and the contour with equation $\mathrm{f}(x, y) = c$ is drawn in the $x - y$ plane, then the vector **grad** f, drawn starting at point A'(the foot of the perpendicular

from A to the $x - y$ plane) is normal to the contour, points in the direction of greatest slope, and has magnitude equal to the greatest gradient at A.


A
gradf
A'

Implicit functions

To find the derivative of an implicit function $f(x,y)$ we note that because $\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}$ is a normal to the curve, considered as a contour, then $\begin{pmatrix} -\partial f/\partial x \\ \partial f/\partial y \end{pmatrix}$ must be tangential and so gives the derivative.

i.e. $\frac{dy}{dx} = -\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}$

Stationary points

A STATIONARY POINT on a surface is a point where the tangent plane is horizontal.
i.e. $z = \text{constant} \Rightarrow \frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero.
It is a local maximum if $f(a+h, b+k) < f(a,b)$ for all sufficiently small h, k (not both zero)
It is a local minimum if $f(a+h, b+k) > f(a,b)$ for all sufficiently small h, k (not both zero)
It is a SADDLE point if $f(a+h, b+k) < f(a,b)$ for all sufficiently small h, k (not both zero)

Functions of more than two variables

If $w = g(x,y,z)$ then we have $\delta w \approx \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z$ which gives an approximation for the change in w due to small changes in x, y and z.

The vector **grad** g is now defined to be $\begin{pmatrix} \partial w/\partial x \\ \partial w/\partial y \\ \partial w/\partial z \end{pmatrix}$ and may be interpreted as the vector whose magnitude and direction gives the greatest rate of change of w and the direction in which it occurs.

The surface $g(x,y,z) = k$

The set of points (x, y, z) for which $w = k$ forms a surface with equation $g(x, y, z) = k$, this is the three-dimensional equivalent of a contour of a function of two variables. It is sometimes known as a CONTOUR SURFACE

If $A(a,b,c)$ is a point of the surface $w = k$, and $\hat{\mathbf{u}}$ is any unit vector tangential to the surface at A, then since w remains constant in the surface, the directional derivative in the direction of $\hat{\mathbf{u}}$ is zero.
i.e. $\hat{\mathbf{u}}$.**grad** g= 0 $\Rightarrow$ **grad** g is perpendicular to $\hat{\mathbf{u}}$ and hence to all vectors in the tangent plane so it is a normal vector for the tangent plane.
The tangent plane thus has equation $\frac{\partial w}{\partial x}x + \frac{\partial w}{\partial y}y + \frac{\partial w}{\partial z}z = \text{constant} = \frac{\partial w}{\partial x}a + \frac{\partial w}{\partial y}b + \frac{\partial w}{\partial z}c$
and equation of normal at A is **r**= **a**+λ**grad** g

Ex A surface has equation $z = x^2y - 2xy - y^2 - 3y + 8$
(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
(ii) Find the co-ordinates of the three stationary points on the surface.
(iii) Find the equation of the tangent plane to the surface at the point $(0, 1, 4)$
(iv) The normal line at the point $(2, 0, 8)$ meets the surface again at the point P.
Find the co-ordinates of P.

(i) $z = x^2y - 2xy - y^2 - 3y + 8 \Rightarrow \frac{\partial z}{\partial x} = 2xy - 2y$ and $\frac{\partial z}{\partial y} = x^2 - 2x - 2y - 3$
(ii) At stationary points $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$ i.e. $xy - y = 0 \Rightarrow y = 0$ or $x = 1$
substituting in $\frac{\partial z}{\partial y} = 0$ we have $x^2 - 2x - 3 = 0$ if $y = 0$ i.e. $(x-3)(x+1) = 0 \Rightarrow x = 3$ or -1
$x = 3,\ y = 0 \Rightarrow z = 8$ and $x = -1, y = 0 \Rightarrow z = 8$
and $y = -2$ if $x = 1 \Rightarrow z = -2 + 4 - 4 + 6 + 8 = 12$ so stationary points are $(-1, 0, 8), (3, 0, 8)$ & $(1, -2, 12)$
(iii) At $(0, 1, 4)$ we have $\frac{\partial z}{\partial x} = -2$ and $\frac{\partial z}{\partial y} = -5$ so tangent plane is $z - 4 = -2(x) - 5(y - 1)$
i.e. $2x + 5y + z = 9$
(iv) Let $g(x, y, z) = x^2y - 2xy - y^2 - 3y - z + 8$

At $(2, 0, 8)$ $\frac{\partial g}{\partial x} = 0$ $\frac{\partial g}{\partial y} = -3$ and $\frac{\partial g}{\partial z} = -1$ so $\begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$ is a normal vector

Equation of normal line is thus $r = \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ which meets surface again when

$12\lambda - 12\lambda - 9\lambda^2 - 9\lambda - (8 + \lambda) + 8 = 0 \Rightarrow 9\lambda^2 + 10\lambda = 0 \Rightarrow \lambda = 0$ or $-\frac{10}{9}$

Co-ordinates of P are thus$(2, 3\times -\frac{10}{9}, 8-\frac{10}{9})$ i.e. $(2, -\frac{10}{3}, \frac{62}{9})$

Ex. (i) Given that $g(x,y,z) = 3x^2 + y^2 + 2z^2 - xy + 2xz - 9$, find $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial z}$

A surface has equation $3x^2 + y^2 + 3z^2 - xy + 2xz - 9 = 0$

(ii) Find the equation of the normal line to the surface at the point $P(1,-1,-2)$

(iii) This normal line meets the surface again at the point Q. Find the co-ordinates of Q, and show that the normal line at P is also the normal line at Q.

(iv) Find the co-ordinates of the two points on the surface where the tangent plane is parallel to the plane $x = 0$

(i) $g(x,y,z) = 3x^2 + y^2 + 2z^2 - xy + 2xz - 9$

$\Rightarrow \frac{\partial g}{\partial x} = 6x - y + 2z,\ \frac{\partial g}{\partial y} = 2y - x$ and $\frac{\partial g}{\partial z} = 4z + 2x$

(ii) at $P(1,-1,-2)$, $\frac{\partial g}{\partial x} = 3$, $\frac{\partial g}{\partial y} = -3$ and $\frac{\partial g}{\partial z} = -6$

Hence, equation of normal is $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

(iii) Parametric equations of normal are $x = t,\ y = -t,\ z = -2t$

and this meets the curve where $3t^2 + t^2 + 8t^2 + t^2 - 4t^2 - 9 = 0 \Rightarrow t = \pm 1$

$t = 1$ corresponds to the given point so the other point Q is $(-1, 1, 2)$

at $Q(-1,1,2)$, $\frac{\partial g}{\partial x} = -3$, $\frac{\partial g}{\partial y} = 3$ and $\frac{\partial g}{\partial z} = 6$ so equation of normal at Q is $\mathrm{r} = s\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

which is clearly the same line as the normal at P.

(iv) tangent plane is parallel to $x = 0$ where $\frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0 \Rightarrow 2y = x$ and $4z = -2x$

i.e. at points of the form $(2k, k, -k)$

such points lie on the surface if $12k^2 + k^2 + 2k^2 - 2k^2 - 4k^2 - 9 = 0 \Rightarrow k = \pm 1$

hence, the required points are $(2, 1, -1)$ and $(-2, -1, 1)$

Differential Geometry

Envelopes

To find the envelope of a system of curves of the form $f(x, y, p) = 0$ where p is a parameter we find the partial derivative of f with respect to the parameter and solve $f(x, y, p) = 0$ and $\frac{\partial f}{\partial p} = 0$ simultaneously.

Arc Length

The "positive sense" along a curve is the senase in which a point moves as the independent variable (or parameter) increases.

By Pythagoras, in an elemental triangle we have $\delta s = \sqrt{(\delta x)^2 + (\delta y)^2}$ hence, $\frac{ds}{dp} = \sqrt{\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2}$

so arc length $= \int_{p=a}^{p=b} \sqrt{\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2}\ dp = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_{y=a}^{y=b} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$

depending on how the equation is defined.

In polar coordinates we have $s = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\ d\theta$

Solids of revolution

Consider a curve $y = f(x)$ rotated through four right angles about the x axis, then we have.

Volume generated between $x = a$ and $x = b$ is $\pi \int_a^b (f(x))^2\ dx$

Curved surface area produced is $\int_A^B 2\pi y\ ds = \int_a^b 2\pi y \frac{ds}{dx}\ dx = \int_{p_1}^{p_2} 2\pi y \frac{ds}{dp}\ dp = \int_{p_1}^{p_2} 2\pi y \sqrt{\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2}\ dp$

Intrinsic equations

The INTRINSIC equation of a curve is the equation connecting the arc length s with the angle ψ (psi) between the tangent to the curve and a fixed direction, usually a horizontal axis.

Curvature

The CURVATURE at a point P is the rate of change of ψ witrh respect to s at P and is denoted by κ i.e. $\kappa = \frac{d\psi}{ds}$. Curve is curving to the left of the positive tangent if $\kappa > 0$.

$\kappa = 0$ at a point of inflection but the converse does not apply.

If the intrinsic equation is known then differentiating with respect to ψ and inverting will give the curvature. If we do not have the intrinsic equation it is rather more difficult.

From cartesian equation, $\kappa = \dfrac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}$ or from parametric equations $\kappa = \dfrac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$

In this last form, the primes denote differentiation with respect to the parameter.

The Circle of curvature

The circle with its centre on the normal through P, which touches the curve at P, has curvature κ and is called the circle of curvature. It's radius, $\rho = \frac{1}{\kappa}$ is the radius of curvature at P.

i.e. radius of curvature $\rho = \dfrac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \dfrac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

Centre of curvature

Let $\hat{\mathbf{t}}$ be the unit vector in the direction of the positive tangent and $\hat{\mathbf{n}}$ the unit vector in the direction of the positive normal, i.e. 90° anticlockwise from $\hat{\mathbf{t}}$.

If C is the centre of curvature then $PC = \rho\hat{\mathbf{n}}$ and the position vector of C is given by $\mathbf{c} = \mathbf{r} + \rho\hat{\mathbf{n}}$

In coordinate form, $x = a - \rho \sin\psi,\ y = b + \rho\cos\psi$ where $C(x, y)$ is the centre of curvature at $P(a, b)$

The evolute of a curve

The EVOLUTE of a curve is the locus of the centre of curvature as P moves along the curve, alternatively it is the envelope of the normals to the curve at the point P. This is usually the easiest way to find it.

Involutes

If a string is wrapped round the evolute and then unwound, keeping the free part taut and straight, then the end of the string describes the original curve, whilst any other point of the string describes a "parallel" curve, these parallel curves are called INVOLUTES of the evolute.

Ex (i) For the point P(0, 1) on the curve $y = e^{2x}$, calulate

(a) the radius of curvature.

(b) the co-ordinates of the centre of curvature.

(ii) (a) Find the equation of the straight line passing through the points $(t, 0)$ and $(0, t^3)$

(b) As t varies, these straight lines define an envelope. Find the cartesian equation of the envelope.

(i) (a) $y = e^{2x} \Rightarrow \frac{dy}{dx} = 2e^{2x}$ and $\frac{d^2y}{dx^2} = 4e^{2x}$ so $\kappa = \frac{4e^{2x}}{(1+4e^{4x})^{3/2}} = \frac{4}{5\sqrt{5}}$ at P

so radius of curvature is $\frac{5\sqrt{5}}{4}$

(b) $\hat{\boldsymbol{t}} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ so $\hat{\boldsymbol{n}} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$ hence centre of curvature is at $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{5\sqrt{5}}{4}\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} -5/2 \\ 9/4 \end{pmatrix}$

(ii) (a) equation of linej oining $(t, 0)$ and $(0, t^3)$ is $y = \frac{t^3}{-t}(x - t) \Rightarrow y = -t^2x + t^3$

(b) Let $g(x, y, t) = y + t^2x - t^3$ then $\frac{\partial g}{\partial t} = 2tx - 3t^2$ so solving $y + t^2x - t^3 = 0$ and $2tx - 3t^2 = 0$ simultaneously we have $t = \frac{2}{3}x$ (since $t \neq 0$) $\Rightarrow y + \frac{4}{9}x^3 - \frac{8}{27}x^3 = 0 \Rightarrow y = -\frac{4}{27}x^3$

Which is the equation of the envelope.

Ex A curve has parametric equations $x = 4t - \frac{1}{3}t^3,\ y = 2t^2 - 8$.

(i) Show that the radius of curvature at a general point $(4t - \frac{1}{3}t^3, 2t^2 - 8.)$ on the curve is $\frac{1}{4}(4 + t^2)^2$

(ii) Find the centre of curvature corresponding to the point on the curve given by $t = 3$.

The arc of the curve given by $0 \le t \le 2\sqrt{3}$ is dsenoted by C.

(iii) Find the length of the arc C.

(iv) Find the area of the curved surface generated when the arc C is rotated about the x-axis.

(i) $x' = 4 - t^2;\ y' = 4t;\ x'' = -2t;\ y'' = 4$

so $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} = \frac{[(4-t^2)^2+16t^2]^{3/2}}{4(4-t^2)-+8t^2} = \frac{(4+t^2)^3}{4(4+t^2)} = \frac{1}{4}(4 + t^2)^2$

(ii) At $t = 3,\ x = 3; y = 10; x' = -5$ and $y' = 12$ so $\frac{dy}{dx} = -\frac{12}{5}$ and $\rho = \frac{169}{4}$

$\hat{\boldsymbol{t}} = \begin{pmatrix} -5/13 \\ 12/13 \end{pmatrix} \Rightarrow \hat{\boldsymbol{n}} = \begin{pmatrix} -12/13 \\ -5/13 \end{pmatrix}$ so centre of curvature is at $\begin{pmatrix} 3 \\ 10 \end{pmatrix} + \frac{169}{4}\begin{pmatrix} -12/13 \\ -5/13 \end{pmatrix} = \begin{pmatrix} -36 \\ -25/4 \end{pmatrix}$

i.e. centre of curvature is at $(-36, -\frac{25}{4})$

(iii) Length of arc is $\int_0^{2\sqrt{3}} \sqrt{x'^2 + y'^2}\, dt = \int_0^{2\sqrt{3}} (4 + t^2)dt = \left[4t + \frac{1}{3}t^3\right]_0^{2\sqrt{3}} = 8\sqrt{3} + 8\sqrt{3} = 16\sqrt{3}$

(iv) Area of curved surface is $2\pi \int_0^{2\sqrt{3}} y\sqrt{x'^2 + y'^2}\, dt = 2\pi \int_0^{2\sqrt{3}} (2t^2 - 8)(4 + t^2)dt$

$= 2\pi \int_0^{2\sqrt{3}} (2t^4 - 32)dt = 2\pi[\frac{2}{5}t^5 - 32t]_0^{2\sqrt{3}} = 2\pi(\frac{2}{5} \times 32 \times 9\sqrt{3} - 64\sqrt{3}) = \frac{512\sqrt{3}\,\pi}{5}$

Abstract Algebra

A "binary operatrion" on a given set of elements, is a rule which takes two elements of the set (which need not be distinct) and forms a definite third element (which may be the same as either of the original ones). Typical examples are:

Addition of numbers, vectors, matrices etc.

Multiplication of numbers or matrices.

Combination of permutations, functions etc.

If the result of the operation is always an element of the original set then we say the operation is CLOSED

If changing the order of the two elements gives the same result, i.e. $a * b = b * a$ then we say the operation is COMMUTATIVE

If $a * (b * c) = (a * b) * c$ for all a, b, c in the set we say the operation is ASSOCIATIVE

If there is an element e in the set such that $a * e = e * a = a$ for every element a in the set then e is called an IDENTITY element for the operation.

If for an element a in the set, there exists an element b such that $a * b = b * a = e$ then we say that b is the INVERSE of a, more usually denoted by a^{-1} unless the operation is one of addition when we use $-a$ for the inverse.

Groups

A non-empty set S together with a binary operation $*$ is said to form a GROUP if the following four axioms hold true.

(C) $*$ is *closed* in S. i.e. a, b in $S \Rightarrow a * b$ and $b * a$ are in S (note that $a * b$ and $b * a$ may be different)

(A) $*$ is *associative*. i.e. $a * (b * c) = (a * b) * c$ for all a, b, c in S.

(N) There is an *identity* or *neutral* element e in S such that $a * e = e * a = a$ for all a in S

(I) Each element a in S has an *inverse* element a^{-1} in S such that $a * a^{-1} = a^{-1} * a = e$

If, in addition, $a * b = b * a$ for every a, b in S then we say it is a COMMUTATIVE or ABELIAN group.

Immediate consequences of the axioms are:

(a) The identity element is unique. If e and f are two identity elements then $ef = e$ and $ef = f \Rightarrow e = f$

(b) Inverses are unique. If a has two inverses, p and q then $p = pe = p(aq) = (pa)q = eq = q$

(c) Cancellation laws hold: $ha = ha \Rightarrow h^{-1}(ha) = h^{-1}(hb) \Rightarrow (h^{-1}h)a = (h^{-1}h)b \Rightarrow a = b$

(d) $ax = b$ has a unique solution: $ax = b \Rightarrow a^{-1}(ax) = a^{-1}b \Rightarrow (a^{-1}a)x = a^{-1}b \Rightarrow x = a^{-1}b$

But note that $ax = b$ does not imply that $x = ba^{-1}$ unless the group is commutative.

The order of a group

A group may be either finite or infinite. A finite group means that it has a definite finite number of elements, the ORDER of the group.

Isomorphisms

A mapping between two groups of equal size which preserves the structure of the groups is called an ISOMORPHISM and the groups are said to be ISOMORPHIC.

A mapping f is an isomorphism if, for any a, b in the first set we have $f(a * b) = f(a) o f(b)$ where $*$ is the operation in the first set and o is the operation in the second set.

Subgroups

If there is a subset of the elements of a group $G = (S, *)$ which has the group properties itself then it is called a SUBGROUP of G.

A PROPER subgroup is any non-empty subgroup excluding the whole group itself. {e} and G are TRIVIAL subgroups. All others are called "NON-TRIVIAL"

Cayley's Theorem

A finite group G of order n is isomorphic to a subgroup of the group of permutations of n elements.

Lagrange's Theorem

This states that the order of any subgroup of a finite group G, must be a divisor of the order of G.

See proof in text book. (page 155)

Cyclic groups

If all the elements of a finite group G are powers of some element of G then it is called a CYCLIC group.

Groups of prime order are necessarily cyclic.

Ex G is a finite group which is commutative. S and T are subgroups of G, and $S \cap T = \{e\}$, where e is

the identity element of G.
We define ST to be the set of all elements of the form st, where $s\in S$ and $t\in T$.
(i) Show that ST is a subgroup of G.
(ii) Show that if $s_1t_1 = s_2t_2$ where $s_1, s_2 \in S$ and $t_1, t_2 \in T$, then $s_1 = s_2$ and $t_1 = t_2$.
(iii) Deduce that if S contains k elements and T contains m elements, then ST contains km elements.
Now let $G = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35\}$, where the binary operation is multiplication .
modulo 36. You may assume that G is a commutative group.
Three cyclic subgroups of G are $S = \{1, 13, 25\}$; $T = \{1, 17\}$ and $U = \{1, 35\}$
(iv) List the elements of the subgroup ST and show that ST is cyclic.
(v) List the elements of TU and show that TU is not cyclic.
(i) $x, y \in ST \Rightarrow x = s_1t_1$ and $y = s_2t_2$ for some $s_1, s_2 \in S$, $t_1, t_2 \in T$
$xy = s_1t_1s_2t_2 = s_1s_2t_1t_2$ (since G is a commutativeg roup)
but $s_1s_2 \in S$ and $t_1t_2 \in T$ so $xy \in ST$ and we have closure.
We are given that ST contains the identity element.
Associativity is inherited from G so ST is a subgroup og G.
(ii) $s_1t_1 = s_2t_2 \Rightarrow s_2^{-1}s_1 = t_2t_1^{-1}$ and $s_2^{-1}s_1 \in S$ and $t_2t_1^{-1} \in T$ so we must have $s_2^{-1}s_1 = t_2t_1^{-1} = e$
$s_2^{-1}s_1 = e \Rightarrow s_1 = s_2$ and similarly $t_1 = t_2$
(iii) From (ii) all the elements st are distinct so we have
$(k-1)$ elements of the form se and $m-1$ of the form
$et + ee + (k-1)(m-1)$ of the form st giving a total of $(k-1)(m-1)+(k-1)+(m-1)+1$ elements , i.e.
Km.
(iv) ST contains 1, 13, 17 and 25 so must also contain 13×17 and 25×17 (mod 36) i.e. 5 and 29
so $ST = \{1, 5, 13, 17, 25, 29\}$ and by inspection this is cyclic, generated by the element 5.
(v) TU contains 1, 17 and 35 so only other element is 17×35 (mod 36) = 19
hence, $TU = \{1, 17, 19, 35\}$ and since every element is self inverse it cannot be cyclic.

Ex. A non-Abelian group G consists of eight 2×2 matrices, and the binary operation is matrix multiplication. The eight distinct elements of G can be _written as $G = \{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \mathbf{A}^3\mathbf{B}\}$
where $\mathbf{I}$ is the identity matrix, and $\mathbf{A}, \mathbf{B}$ are 2×2 matrices_such that $\mathbf{A}^4 = \mathbf{I}$, $\mathbf{B}^2 = \mathbf{I}$ and $\mathbf{BA} = \mathbf{A}^3\mathbf{B}$
(i) Show that $(\mathbf{A}^2\mathbf{B})(\mathbf{AB}) = \mathbf{A}$ and $(\mathbf{AB})(\mathbf{A}^2\mathbf{B}) = \mathbf{A}^3$
(ii) Evaluate the following products, giving each one as an element of G as listed above.
$\mathbf{(AB)(A)}$, $\mathbf{(AB)(AB)}$, $\mathbf{(B)(A^3)}$
(iii) Find the order of each element of G.
(iv) Show that
(v) Find the other two subgroups of G which have order 4.
(vi) For each of the three subgroups of order 4, state whether or not it is a cyclic subgroup.

Solution

(i) $(\mathbf{A}^2\mathbf{B})(\mathbf{AB}) = \mathbf{A}^2(\mathbf{BA})\mathbf{B} = \mathbf{A}^2(\mathbf{A}^3\mathbf{B})\mathbf{B} = \mathbf{A}^5\mathbf{B}^2 = \mathbf{A}^4\mathbf{AB}^2 = \mathbf{IAI} = \mathbf{A}$
$(\mathbf{AB})(\mathbf{A}^2\mathbf{B}) = \mathbf{A}(\mathbf{BA})\mathbf{AB} = \mathbf{A}(\mathbf{A}^3\mathbf{B})\mathbf{AB} = \mathbf{A}^4\mathbf{BAB} = \mathbf{IA}^3\mathbf{B}^2 = \mathbf{A}^3\mathbf{I} = \mathbf{A}^3$
(ii) $(\mathbf{AB})(\mathbf{A}) = \mathbf{A}(\mathbf{BA}) = \mathbf{AA}^3\mathbf{B} = \mathbf{IB} = \mathbf{B}$
$(\mathbf{AB})(\mathbf{AB}) = \mathbf{A}(\mathbf{BA})\mathbf{B} = \mathbf{A}(\mathbf{A}^3\mathbf{B})\mathbf{B} = \mathbf{A}^4\mathbf{B}^2 = \mathbf{II} = \mathbf{I}$
$(\mathbf{B})(\mathbf{A}^2) = (\mathbf{BA})\mathbf{A} = \mathbf{A}^3\mathbf{BA} = \mathbf{A}^3\mathbf{A}^3\mathbf{B} = \mathbf{A}^6\mathbf{B} = \mathbf{A}^2\mathbf{B}$
(iii) $\mathbf{A}$ and $\mathbf{A}^3$ are of order 4, $\mathbf{A}^2$, $\mathbf{B}$ and $\mathbf{AB}$ are of order 2
$(\mathbf{A}^2\mathbf{B})(\mathbf{A}^2\mathbf{B}) = \mathbf{A}^2(\mathbf{BA})(\mathbf{AB}) = \mathbf{A}^2\mathbf{A}^3\mathbf{BAB} = \mathbf{ABAB} = \mathbf{I}$ so $\mathbf{A}^2\mathbf{B}$ is of order 2
$(\mathbf{A}^3\mathbf{B})(\mathbf{A}^3\mathbf{B}) = \mathbf{A}^3\mathbf{A}^3\mathbf{AB}^2\mathbf{B} = \mathbf{A}^2\mathbf{BA}^2\mathbf{B} = \mathbf{I}$ so $\mathbf{A}^3\mathbf{B}$ has order 2
Obviously $\mathbf{I}$ has order 1
(iv) Using results already established we can construct the table for $\{\mathbf{I}, \mathbf{A}^2, \mathbf{B}, \mathbf{A}^2\mathbf{B}\}$

	$\mathbf{I}$	$\mathbf{A}^2$	$\mathbf{B}$	$\mathbf{A}^2\mathbf{B}$
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{A}^2$	$\mathbf{B}$	$\mathbf{A}^2\mathbf{B}$
$\mathbf{A}^2$	$\mathbf{A}^2$	$\mathbf{I}$	$\mathbf{A}^2\mathbf{B}$	$\mathbf{B}$
$\mathbf{B}$	$\mathbf{B}$	$\mathbf{A}^2\mathbf{B}$	$\mathbf{I}$	$\mathbf{A}^2$
$\mathbf{A}^2\mathbf{B}$	$\mathbf{A}^2\mathbf{B}$	$\mathbf{B}$	$\mathbf{A}^2$	$\mathbf{I}$

From the table we see that we have closure, an identity and every element has an inverse.
Associativity is inherited from G hence $\{\mathbf{I}, \mathbf{A}^2, \mathbf{B}, \mathbf{A}^2\mathbf{B}\}$ is a subgroup of G
(iv) other two subgroups of order 4 are $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3\}$

and $\{\mathbf{I}, \mathbf{A}^2, \mathbf{A}^3\mathbf{B}, \mathbf{AB}\}$

The first is obviously a cyclic group of order 4

and we verify the other by constructing a table

verifying that it is a subgroup

(vi) $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3\}$ is cyclic, the other two are not

	$\mathbf{I}$	$\mathbf{A}^2$	$\mathbf{AB}$	$\mathbf{A}^3\mathbf{B}$
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{A}^2$	$\mathbf{AB}$	$\mathbf{A}^3\mathbf{B}$
$\mathbf{A}^2$	$\mathbf{A}^2$	$\mathbf{I}$	$\mathbf{A}^3\mathbf{B}$	$\mathbf{AB}$
$\mathbf{AB}$	$\mathbf{AB}$	$\mathbf{A}^3\mathbf{B}$	$\mathbf{I}$	$\mathbf{A}^2$
$\mathbf{A}^3\mathbf{B}$	$\mathbf{A}^3\mathbf{B}$	$\mathbf{AB}$	$\mathbf{A}^2$	$\mathbf{I}$

Markov Chains

A MARKOV CHAIN is a sequence of events where the probability of an outcome at one stage depends only on the outcome of the event at the previous stage.

A TRANSITION matrix consists of the TRANSITION PROBABILITIES (The probabilities of passing from one stage to the next) Each column of the transition matrux is a PROBABILITY VECTOR

If **P** is the transition matrix and **p** a particular probability vector representing the probabilities at one stage then **Pp** represents the probabilities at the next stage, $\mathbf{P}^2\mathbf{p}$ the probabilities at the next but one stage and so on.

Powers of the transition matrix give conditional probabilities of moving from one outcome to another over 2, 3, 4, ... steps.

Ex. A study of the weather over a long period of time suggests that if today is sunny then there is a 70% chance that tomorrow will also be sunny whilst if today is cloudy then there is an 80% chance that tomorrow will also be cloudy. It is sunny on a particular Saturday, what is the chance that it will be

(a) sunny on the following Monday

(b) cloudy on the following Tuesday.

(c) sunny on the following Wednesday.

We first form a transition matrix $\mathbf{P} = \begin{pmatrix} & \text{sunny} & \text{cloudy} \\ \text{sunny} & 0.7 & 0.2 \\ \text{cloudy} & 0.3 & 0.8 \end{pmatrix}$

(a) Monday involves two steps so probabilities are given by $\mathbf{P}^2 = \begin{pmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{pmatrix}$

so chance that it is sunny on Monday is 0.55

(b) Tuesday involves three step so we require $\mathbf{P}^3 = \begin{pmatrix} 0.475 & 0.35 \\ 0.525 & 0.65 \end{pmatrix}$

so chance that it is cloudy on Tuesday is 0.525

(c) Finally, Wednesday involves four steps so $\mathbf{P}^4 = \begin{pmatrix} 0.4375 & 0.375 \\ 0.5625 & 0.625 \end{pmatrix}$

so chance that it is sunny on Wednesday is 0.4375

If a Markov chain has more that two states at each stage then the transition matyrix will be larger e.g. 3×3 for three possible outcomes at each stage and so on but the general principle remains the same.

It is for this reason that you are advised that a calculator that will handle matrix multiplication is essential when studying this topic.

For certain transition matrices **P** we find that as $n \to \infty$, $\mathbf{P}^n \to$ a matrix whose columns are identical. These columns give the EQUILIBRIUM PROBABILITIES and show that, in the long run, the probabilities of each state are independent of what happened at the beginning.

In the example above we find that the equilibrium matrix is $\begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$

i.e. this means that there is a 40% chance of a sunny day at any time in the distant future.

This matrix can be found either by iteration i.e. Raising **P** to higher and higher powers, or as follows

If $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ is the equilibrium vector then we must have $\mathbf{Pp} = \mathbf{p}$

Again using the same example as before this gives $0.7p_1 + 0.2p_2 = p_1$ and $0.3p_1 + 0.8p_2 = p_2$

i.e. $0.2p_2 - 0.3p_1 = 0$ and $0.3p_1 - 0.2p_2 = 0$ also of course $p_1 + p_2 = 1$ so eliminating p_2 we have

$0.2 - 0.2p_1 - 0.3p_1 = 0 \Rightarrow 0.5p_1 = 0.2 \Rightarrow p_1 = 0.4$ and $p_2 = 0.6$

Run length of a Markov chain

Still using our sunny/cloudy example, if it is sunny on Monday, what is the probability that it is sunny for the rest of the week, i.e. up to and including Saturday but cloudy on Sunday.

This is easily found to be $0.7^5 \times 0.30 = 0.050421$

Generalising, the probability that it is sunny for the next X days is $\mathrm{P}(X = r) = 0.7^r \times 0.3$

i.e. the probabilities are the terms of a geometric sequence. So we can easily find the expected run length by evaluating $\sum_1^\infty r\mathrm{P}(X = r) = 0.3 \times 0.7 + 2 \times 0.3 \times 0.7^2 + 3 \times 0.3 \times 0.7^3 + \ldots$

which is 0.3×0.7 times the binomial expansion of $(1 - 0.7)^{-2}$

hence, expected number of further sunny days after a sunny day is $\frac{0.3 \times 0.7}{0.3^2} = \frac{0.21}{0.09} = 2.33$

Generalising, if a is the probability that the system remains in the same state at the next stage then the expected run length for that state is $\frac{a(1-a)}{(1-a)^2} = \frac{a}{1-a}$

Note that this is the expected (mean) number of FURTHER consecutive days that the initial state remains unchanged and is one less than the number of times the state is repeated.

Classification of Markov Chains

A Markov chain is said to be REGULAR if some power of the transition matrix contains only positive entries. In a regular Markov chain it is possible to pass from any state to any other state and there is a unique limiting probability vector.

Random Walks

If it is possible to assign an order to the various states so that from any one state it is only possible to move to a limited number of other states then this can be described as a RANDOM WALK.

Periodic chains

If $\mathbf{P}$ is a transition matrix and for some value of n $\mathbf{P}^{n+1} = \mathbf{P}$ then the Markov chain is PERIODIC with PERIOD n.

Ex. $\mathbf{P} = \begin{pmatrix} 0. & 0.5 & 0 \\ 1 & 0 & 1 \\ 0 & 0.5 & 0 \end{pmatrix} \Rightarrow \mathbf{P}^2 = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix}$ and $\mathbf{P}^3 = \begin{pmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 1 \\ 0 & 0.5 & 0 \end{pmatrix} = \mathbf{P}$

Note that $\mathbf{P}^{2n} = \mathbf{P}^n$ and $\mathbf{P}^{2n-1} = \mathbf{P}^{2n+1}$ for all n so periodic with period 2

Reflecting Barriers

If at some stage, the outcome is inevitable then we have a REFLECTING BARRIER.

Ex. The example above represents the following situation. A bag contains 2 balls, both of which are black or white.

A ball is drawn at random and replaced with one of the opposite colour. The transition matrix for the number of black balls in the bag at each stage is $\mathbf{P} = \begin{pmatrix} & 0b & 1b & 2b \\ 0b & 0 & 0.5 & 0 \\ 1b & 1 & 0 & 1 \\ 2b & 0 & 0.5 & 0 \end{pmatrix}$

As we saw above if there are 0 black balls in the bag then at the next stage there must be one and similarly if there were 2 black balls in the bag then at the next stage there must be only one.

We therefore have two reflecting barriers. The states of 0 black balls or 2 black balls.

A zero in the leading diagonal of a transition matrix with a 1 above or below it indicates a reflecting barrier.

Absorbing States

If at some stage we arrive at some state where it is impossible to leave that state then we have an ABSORBING BARRIER.

Ex. A scientist is studying a colony of a particular organism, and classifies the individuals as being in one of three states, well, ill or dead. Which of these is an absorbing state?

If at some stage the organism is dead then it cannot change that state so this is the absorbing state.

Ex. A decorative light glows in one of four colours, purple,blue,red and yellow. The colour changes at the end of a fixed time interval and the transition matrix is as follows:

$$\begin{array}{cc} & \begin{array}{cccc} P & B & R & Y \end{array} \\ \begin{array}{c} P \\ B \\ R \\ Y \end{array} & \begin{pmatrix} 0 & 0.2 & 0.4 & 0 \\ 1 & 0 & 0.2 & 0 \\ 0 & 0.4 & 0 & 1 \\ 0 & 0.4 & 0.4 & 0 \end{pmatrix} \end{array}$$

(i) Explain the significance of the zeros in the leading diagonal of the matrix.

(ii) Which colour occurs most often in the long run? What proportion of the time does the light show that colour?

The light develops a fault. Once it shows blue it gets stuck there and can no longer change to any other colour.

(iii) Write down the new transition matrix and describe any features that the chain now has.

(iv) When the fault occurs, the light is showing red. What is the probability that seven intervals later it is still not showing blue?

(i) The zeros show that the light never shows the same colour in two successive intervals.

$$\begin{array}{cc} & \begin{array}{cccc} P & B & R & Y \end{array} \\ \begin{array}{c} P \\ B \\ R \\ Y \end{array} & \begin{pmatrix} 0 & 0.2 & 0.4 & 0 \\ 1 & 0 & 0.2 & 0 \\ 0 & 0.4 & 0 & 1 \\ 0 & 0.4 & 0.4 & 0 \end{pmatrix} \end{array} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \Rightarrow 0.2p_2 + 0.4p_3 = p_1,\ p_1 + 0.2p_3 = p_2,\ 0.4p_2 + p_4 = p_3,$$

(ii) $0.4p_2 + 0.4p_3 = p_4$ so $0.2p_2 + 0.4p_3 = p_2 - 0.2p_3 \Rightarrow 0.6p_3 = 0.8p_2$ so $3p_3 = 4p_2$

hence, $p_1 = 0.55p_3$ and $p_4 = 0.7p_3$ so $p_1 : p_2 : p_3 : p_4 = 0.55 : 0.75 : 1 ; 0.7$ so p_3 is the greatest

i.e. red occurs most frequently ,with probability $\frac{1}{3}$

(iii) $\mathbf{M} = \begin{array}{cc} & \begin{array}{cccc} P & B & R & Y \end{array} \\ \begin{array}{c} P \\ B \\ R \\ Y \end{array} & \begin{pmatrix} 0 & 0 & 0.4 & 0 \\ 1 & 1 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.4 & 0 \end{pmatrix} \end{array}$ Blue is an absorbing state.

(iv) $\mathbf{M}^7 = \begin{array}{cc} & \begin{array}{cccc} P & B & R & Y \end{array} \\ \begin{array}{c} P \\ B \\ R \\ Y \end{array} & \begin{pmatrix} 0 & 0 & 0.0256 & 0 \\ 1 & 1 & 0.9488 & 0.936 \\ 0 & 0 & 0 & 0.064 \\ 0 & 0 & 0.0256 & 0 \end{pmatrix} \end{array}$ so probability of blue after 7 intervals is 0.9488 and so the probability of not showing blue is 0.0512