

Question 27

Solve each of the following trigonometric equations.

a) $\sin\left(\theta + \frac{\pi}{4}\right) = \sin \theta, \quad 0 \leq \theta < 2\pi$

b) $\cos\left(x + \frac{\pi}{6}\right) = \cos\left(x + \frac{2\pi}{3}\right), \quad 0 \leq x < 2\pi$

c) $\sin\left(\frac{\pi}{3} - y\right) = \cos\left(y + \frac{5\pi}{6}\right), \quad 0 \leq y < 2\pi \text{ (very hard)}$

d) $2\cos\left(\varphi + \frac{\pi}{2}\right) + \sin\left(\varphi + \frac{\pi}{3}\right) = 0, \quad 0 \leq \varphi < 2\pi$

e) $\sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{6}\right), \quad 0 \leq \alpha < 2\pi$

$\boxed{\theta = \frac{3\pi}{8}, \frac{11\pi}{8}}, \boxed{x = \frac{7\pi}{12}, \frac{19\pi}{12}}, \boxed{y = \frac{\pi}{2}, \frac{3\pi}{2}}, \boxed{\varphi = \frac{\pi}{6}, \frac{7\pi}{6}}, \boxed{\alpha = \frac{\pi}{12}, \frac{13\pi}{12}}$

(a) $\sin\left(\theta + \frac{\pi}{4}\right) = \sin \theta$

$$\Rightarrow \sin\left(\theta + \frac{\pi}{4}\right) = \sin\left(\theta + 0\right)$$

$$\Rightarrow \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4} = \sin\theta$$

$$\Rightarrow \sin\theta\frac{\sqrt{2}}{2} + \cos\theta\frac{\sqrt{2}}{2} = \sin\theta$$

$$\Rightarrow \sqrt{2}\sin\theta + \sqrt{2}\cos\theta = 2\sin\theta$$

$$\Rightarrow \sqrt{2}\sin\theta + \sqrt{2}\cos\theta = 2\cos\theta$$

$$\Rightarrow \sqrt{2} = (2-\sqrt{2})\tan\theta$$

$$\Rightarrow \tan\theta = \frac{\sqrt{2}}{2-\sqrt{2}}$$

$$\Rightarrow \tan\theta = \frac{\sqrt{2}}{\sqrt{2}(1-\frac{\sqrt{2}}{2})}$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{2}-1}$$

$$\Rightarrow \theta = \frac{\pi}{4} + k\pi, \quad k=0,1,2,3,...$$

$$y_1 = \frac{\pi}{4}, \quad y_2 = \frac{5\pi}{4}$$

(c) $\sin\left(\frac{\pi}{3} - y\right) = \cos\left(y + \frac{5\pi}{6}\right)$

$$\Rightarrow \sin\frac{\pi}{3}\cos y - \cos\frac{\pi}{3}\sin y = \cos y\cos\frac{5\pi}{6} - \sin y\sin\frac{5\pi}{6}$$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos y - \frac{1}{2}\sin y = -\frac{\sqrt{3}}{2}\cos y - \frac{1}{2}\sin y$$

$$\Rightarrow \sqrt{3}\cos y - \sin y = -\sqrt{3}\cos y - \sin y$$

$$\Rightarrow \sqrt{3}\cos y - \sin y = -\sqrt{3}\cos y - \sin y$$

$$\Rightarrow \tan y = \pm\sqrt{3}$$

$$\tan y = \infty$$

$$\arctan(\infty) = \frac{\pi}{2}$$

$$y_1 = \frac{\pi}{6}, \quad y_2 = \frac{5\pi}{6}$$

(e) $\sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{6}\right)$

$$\Rightarrow \sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{6}\right)$$

$$\Rightarrow \sqrt{2}\cos\alpha\cos\frac{\pi}{4} - \sqrt{2}\sin\alpha\sin\frac{\pi}{4} = \sin\alpha\cos\frac{\pi}{6} + \cos\alpha\sin\frac{\pi}{6}$$

$$\Rightarrow -2\sin\alpha + \frac{1}{2}\sin\alpha + \frac{\sqrt{2}}{2}\cos\alpha = 0$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos\alpha = \frac{3}{2}\sin\alpha$$

$$\Rightarrow \sqrt{2}\cos\alpha = 3\sin\alpha$$

$$\Rightarrow \frac{\sqrt{2}}{3} = \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \tan\alpha = \frac{\sqrt{2}}{3}$$

$$\arctan\left(\frac{\sqrt{2}}{3}\right) = \frac{\pi}{12}$$

$$\alpha = \frac{\pi}{12} + k\pi, \quad k=0,1,2,3,...$$

$$y_1 = \frac{\pi}{12}, \quad y_2 = \frac{13\pi}{12}$$

(d) $2\cos\left(\varphi + \frac{\pi}{2}\right) + \sin\left(\varphi + \frac{\pi}{3}\right) = 0$

$$\Rightarrow 2\cos\left(\varphi + \frac{\pi}{2}\right) + 2\sin\left(\varphi + \frac{\pi}{3}\right) + \sin\left(\varphi + \frac{\pi}{3}\right) + \cos\left(\varphi + \frac{\pi}{2}\right) = 0$$

$$\Rightarrow -2\sin\varphi + \frac{1}{2}\sin\varphi + \frac{\sqrt{2}}{2}\cos\varphi = 0$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos\varphi = \frac{3}{2}\sin\varphi$$

$$\Rightarrow \sqrt{2}\cos\varphi = 3\sin\varphi$$

$$\Rightarrow \frac{\sqrt{2}}{3} = \frac{\sin\varphi}{\cos\varphi}$$

$$\Rightarrow \tan\varphi = \frac{\sqrt{2}}{3}$$

$$\arctan\left(\frac{\sqrt{2}}{3}\right) = \frac{\pi}{12}$$

$$\varphi = \frac{\pi}{12} + k\pi, \quad k=0,1,2,3,...$$

$$y_1 = \frac{\pi}{12}, \quad y_2 = \frac{13\pi}{12}$$

(e) $\sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{6}\right)$

$$\Rightarrow \sqrt{2}\cos\alpha\cos\frac{\pi}{4} - \sqrt{2}\sin\alpha\sin\frac{\pi}{4} = \sin\alpha\cos\frac{\pi}{6} + \cos\alpha\sin\frac{\pi}{6}$$

$$\Rightarrow \cos\alpha - \sin\alpha = \frac{\sqrt{2}}{2}\sin\alpha + \frac{1}{2}\cos\alpha$$

$$\Rightarrow 2\cos\alpha - 2\sin\alpha = \sqrt{2}\sin\alpha + \cos\alpha$$

$$\Rightarrow \cos\alpha = (\sqrt{2}+1)\sin\alpha$$

$$\Rightarrow 1 = (\sqrt{2}+2)\frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \tan\alpha = \frac{1}{\sqrt{2}+2}$$

$$\arctan\left(\frac{1}{\sqrt{2}+2}\right) = \frac{\pi}{12}$$

$$\alpha = \frac{\pi}{12} + k\pi, \quad k=0,1,2,3,...$$

$$y_1 = \frac{\pi}{12}, \quad y_2 = \frac{13\pi}{12}$$