Question 1

The following convergent series C and S are given by

$$C = 1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta ...$$

$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta ...$$

a) Show clearly that

$$C + \mathrm{i}S = \frac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}.$$

b) Hence show further that

$$C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta},$$

and find a similar expression for S.

$$\boxed{S = \frac{2\sin\theta}{5 - 4\cos\theta}}$$

(a) $C + iS = 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta) + ...$

$= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i(2\theta)} + \frac{1}{8}e^{i(3\theta)} + ...$

$= \underbrace{1 + \frac{e^{i\theta}}{2} + \left(\frac{e^{i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta}}{2}\right)^3 + ...}_{\text{G.P. with } a=1,\ r=\frac{e^{i\theta}}{2}}$

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{1 - \frac{e^{i\theta}}{2}} = \frac{2}{2 - e^{i\theta}}$ // as required

(b) $C + iS = \frac{2}{2 - e^{i\theta}} = \frac{2(2 - e^{-i\theta})}{(2 - e^{i\theta})(2 - e^{-i\theta})} = \frac{2(2 - (\cos\theta - i\sin\theta))}{4 - 2e^{-i\theta} - 2e^{i\theta} + 1}$

$= \frac{2(2 - \cos\theta + i\sin\theta)}{5 - 2(e^{i\theta} + e^{-i\theta})} = \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cosh i\theta}$

$= \frac{(4 - 2\cos\theta) + i(2\sin\theta)}{5 - 4\cos\theta}$

$\therefore C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}$ // & $S = \frac{2\sin\theta}{5 - 4\cos\theta}$ //

Question 2

The following finite sums, C and S, are given by

$$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$$
$$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$$

By considering the binomial expansion of $(1+A)^5$, show clearly that

$$C = 32\cos^5\theta\cos 5\theta,$$

and find a similar expression for S

$$\boxed{S = 32\cos^5\theta\sin 5\theta}$$

$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$

$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$

THUS

$C + iS = 1 + 5e^{i2\theta} + 10e^{i4\theta} + 10e^{i6\theta} + 5e^{i8\theta} + e^{i10\theta}$

WHICH IS THE BINOMIAL EXPANSION

$= (1 + e^{2i\theta})^5$

$= (1 + \cos 2\theta + i\sin 2\theta)^5$

$= (\cancel{1} + 2\cos^2\theta \cancel{- 1} + 2i\sin\theta\cos\theta)^5$

$= (2\cos^2\theta + i2\sin\theta\cos\theta)^5$

$= [2\cos\theta(\cos\theta + i\sin\theta)]^5$

$= 32\cos^5\theta(\cos\theta + i\sin\theta)^5$

$= 32\cos^5\theta(\cos 5\theta + i\sin 5\theta)$

$= (32\cos^5\theta\cos 5\theta) + i(32\cos^5\theta\sin 5\theta)$

$\therefore\ C = 32\cos^5\theta\cos 5\theta$

$S = 32\cos^5\theta\sin 5\theta$